# SLOCC invariant and semi-invariants for SLOCC classification of four-qubits [1]

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Abstract

We show there are at least 28 distinct true SLOCC entanglement classes for four-qubits by means of SLOCC invariant and semi-invariants and derive the number of the degenerated SLOCC classes for $n$-qubits.

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# 1 Introduction

Entanglement plays a key role in quantum computing and quantum information. If two states can be obtained from each other by means of local operations and classical communication (LOCC) with nonzero probability, we say that two states have the same kind of entanglement[1]. It is well known that a pure entangled state of two-qubits can be locally transformed into a state $GHZ$. For multipartite systems, there are several inequivalent forms of entanglement under asymptotic LOCC[2]. Recently, many authors have investigated the equivalence classes of three-qubit states specified SLOCC (stochastic local operations and classical communication ) [3]$-$[15]. Dür et al. showed that for pure states of three-qubits there are four different degenerated SLOCC entanglement classes and two inequivalent true entanglement classes[4]. A. Miyake discussed the onionlike classification of SLOCC orbits and proposed the SLOCC equivalence classes using the orbits[10]. A.K. Rajagopal and R.W. Rendell gave the conditions for the full separability and the biseparability[14]. In [15] we gave the simple criteria for the complete SLOCC classification for three-qubits and the criteria of a few classes for four-qubits.

Verstraete et al.[9] considered the entanglement classes of four-qubits under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement and claimed that by determinant one SLOCC operations, a pure state of four qubits can be transformed into one of the nine families of states. Clearly, this does not say that each family is a SLOCC class. Then, how many SLOCC classes are there for each family? what are representations? After investigating the nine families by means of methods in this paper, we can say that each of the first six families includes several SLOCC classes. We list SLOCC classes of some families as follows.

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For Family $L_{ab_3}$ : $a$ $(|0000\rangle + |1111\rangle) + \frac{a+b}{2}( \ |0101\rangle + |1010\rangle) + \ \frac{a-b}{2}( \ |0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$, this consists of five true SLOCC entanglement classes, which are $a = b = 0$, i.e. class $|W\rangle$, $a = b \neq 0$, $a = -b \neq 0$, $a \neq \pm b \wedge 3a^2 + b^2 \neq 0$ and $a \neq \pm b \wedge 3a^2 + b^2 = 0$, respectively. For Family $L_{a_4}$ : $a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + (i|0001\rangle + |0110\rangle - i|1011\rangle)$, this includes two true SLOCC entanglement classes: $a = 0$ and $a \neq 0$. For Family $L_{a_2 0_{3\oplus 1}}$ : $a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle$, it includes two SLOCC classes. When $a = 0$, this becomes a product state of a qubit state and 3-qubit $|W\rangle$. When $a \neq 0$, this is a true entangled state.

In [17][18], the authors utilized the partition for SLOCC classification of three-qubits and four-qubits. The idea for the partition was early used to analyze the separability of $n$-qubits and multipartite pure states in [19]. In [18], the authors declared there are eight Span classes and 16 true SLOCC entanglement classes for four-qubits. By means of methods in this paper, we can illustrate how many true SLOCC entanglement classes there are for each Span {.....}. For example, for Span $\{O_k\Psi, O_k\Psi\}$, canonical states are $|0000\rangle + |1100\rangle + a|0011\rangle + b|1111\rangle$ and $|0000\rangle + |1100\rangle + a|0001\rangle + a|0010\rangle + b|1101\rangle + b|1110\rangle$, where $a \neq b$ [18]. It was not pointed out in [18] that what relation $a$ and $b$ satisfy to be a representation of SLOCC class. It can be shown that for the former canonical state, $a = -b$ and $a \neq -b$ represent two true SLOCC classes, respectively while for the latter canonical state, $ab = 0$ and $ab \neq 0$ represent two true SLOCC classes, respectively. We can also explain that each of Span $\{000, GHZ\}$, Span $\{0_i\Psi, 0_j\Psi\}$ and Span $\{GHZ, W\}$ includes four true SLOCC entanglement classes and Span $\{0_k\Psi, GHZ\}$ includes more. Also considering Span $\{000, 000\}$, Span $\{000, 0_k\Psi\}$ and Span $\{000, W\}$, in total, the eight Spans {...} in [18] include much more true SLOCC entanglement classes.

In this paper, we find the SLOCC invariant and semi-invariants for four-qubits. Using the invariant and semi-invariants, we can determine if two states belong to different SLOCC entanglement classes. We distinguish 28 distinct true entanglement classes, where permutations of the qubits are allowed. This classification is not complete. It seems that there are more true entanglement classes. The invariant and semi-invariants only require simple arithmetic operations.

# 2 SLOCC invariant and semi-invariants

We discuss the system comprising four qubits A, B, C and D. The states of a four-qubit system can be generally expressed as

$$|\psi\rangle = \sum_{i=0}^{15} a_i |i\rangle. \tag{1}$$

Two states $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under SLOCC if and only if there exist invertible local operators $\alpha, \beta$, $\gamma$ and $\delta$ such that

$$|\psi\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta |\psi'\rangle, \tag{2}$$

where the local operators $\alpha, \beta$, $\gamma$ and $\delta$ can be expressed as $2 \times 2$ invertible matrices

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix},\ \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}, \delta = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}. \tag{3}$$

## 2.1 SLOCC invariant

Let $|\psi'\rangle = \sum_{i=0}^{15} b_i |i\rangle$ in Eq. (2). If $|\psi\rangle$ is SLOCC equivalent to $|\psi'\rangle$, then the following equation holds.

$$IV(\psi) = IV(\psi') \det(\alpha) \det(\beta) \det(\gamma) \det(\delta), \tag{4}$$

where

$$IV(\psi) = (a_2 a_{13} - a_3 a_{12}) + (a_4 a_{11} - a_5 a_{10}) - (a_0 a_{15} - a_1 a_{14}) - (a_6 a_9 - a_7 a_8) \tag{5}$$

and

$$IV(\psi') = (b_2 b_{13} - b_3 b_{12}) + (b_4 b_{11} - b_5 b_{10}) - (b_0 b_{15} - b_1 b_{14}) - (b_6 b_9 - b_7 b_8). \tag{6}$$

Eq. (4) was by induction derived in [20]. We can also verify Eq. (4) as follows. By solving the matrix equation in Eq. (2), we obtain amplitudes $a_i$ of state $|\psi\rangle$ in Eq. (2). Then substituting $a_i$ into $IV(\psi)$, we have Eq. (4).

Notice that $IV(\psi)$ does not vary under determinant one SLOCC operations ($SL$-operations) or vanish under non-unit-determinant SLOCC operations.

If $\psi$ is $SL$-equivalent to $\psi'$, then $IV(\psi) = IV(\psi')$. Eq. (4) implies that each SLOCC class has infinite $SL$-classes. For family $L_{a_2 0_{3\oplus 1}}$ in [9], let $|\psi'\rangle$ be a representative state of the family: $a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle$. Eq. (4) becomes $IV(\psi) = -a^2 \det(\alpha) \det(\beta) \det(\gamma) \det(\delta)$. For $SL$-operations, $IV(\psi) = IV(\psi') = -a^2$. It is clear that different $a'$s values yield different $SL$-classes. Therefore there are infinite $SL$-classes when $a \neq 0$. However, the infinite $SL$-classes belong to a true SLOCC entanglement class.

## 2.2 Semi-invariants $F_i$

Coffman defined the concurrence of three-qubits. We extend the definition of the concurrence of three-qubits to the one of four-qubits as follows. For state $|\psi\rangle$, we define $F_i(\psi)$ as follows. Notice that $F_3(\psi)$ to $F_8(\psi)$ can be obtained from $F_1(\psi)$ and $F_2(\psi)$ by permutations of the qubits.

$F(\psi) = 4 \sum_{i=1}^{10} |F_i(\psi)|$, where

$F_1(\psi) = (a_0 a_7 - a_2 a_5 + (a_1 a_6 - a_3 a_4))^2 - 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7)$,

$F_2(\psi) = ((a_8a_{15} - a_{11}a_{12}) + (a_9a_{14} - a_{10}a_{13}))^2 - 4(a_{11}a_{13} - a_9a_{15})(a_{10}a_{12} - a_8a_{14})$,

$F_3(\psi) = (a_0a_{11} - a_2a_9 + a_1a_{10} - a_3a_8)^2 - 4(a_2a_8 - a_0a_{10})(a_3a_9 - a_1a_{11})$,

$F_4(\psi) = (a_4a_{15} - a_6a_{13} + a_5a_{14} - a_7a_{12})^2 - 4(a_6a_{12} - a_4a_{14})(a_7a_{13} - a_5a_{15})$,

$F_5(\psi) = (a_0a_{13} - a_4a_9 + a_1a_{12} - a_5a_8)^2 - 4(a_4a_8 - a_0a_{12})(a_5a_9 - a_1a_{13})$,

$F_6(\psi) = (a_2a_{15} - a_6a_{11} + a_3a_{14} - a_7a_{10})^2 - 4(a_6a_{10} - a_2a_{14})(a_7a_{11} - a_3a_{15})$,

$F_7(\psi) = (a_0a_{14} - a_4a_{10} + a_2a_{12} - a_6a_8)^2 - 4(a_4a_8 - a_0a_{12})(a_6a_{10} - a_2a_{14})$,

$F_8(\psi) = (a_1a_{15} - a_5a_{11} + a_3a_{13} - a_7a_9)^2 - 4(a_5a_9 - a_1a_{13})(a_7a_{11} - a_3a_{15})$,

$F_9(\psi) = ((a_0a_{15} - a_2a_{13}) + (a_1a_{14} - a_3a_{12}))^2 - 4(a_0a_{14} - a_2a_{12})(a_1a_{15} - a_3a_{13})$,

$F_{10}(\psi) = ((a_4a_{11} - a_7a_8) + (a_5a_{10} - a_6a_9))^2 - 4(a_7a_9 - a_5a_{11})(a_6a_8 - a_4a_{10}))$.

For state $|\psi'\rangle$, let $F_i(\psi')$ be obtained from $F_i(\psi)$ by replacing $a$ in $F_i(\psi)$ by $b$. Then by induction we can show that $F_i$ have the following interesting properties and the properties are called as semi-invariants.

In Eq. (2), let $\alpha = I$, where $I$ is an identity. Thus, Eq. (2) becomes

$$|\psi\rangle = I \otimes \beta \otimes \gamma \otimes \delta |\psi'\rangle. \tag{7}$$

Then we have the following.

$$F_i(\psi) = F_i(\psi') \overset{2}{\det}(\beta) \overset{2}{\det}(\gamma) \overset{2}{\det}(\delta), i = 1, 2. \tag{8}$$

Eq. (8) can be verified as follows. We obtain amplitudes $a_i$ of state $|\psi\rangle$ by solving Eq. (7). Then substituting $a_i$ into $F_i(\psi)$, we derive Eq. (8).

Also, in Eq. (2), let $\beta = I$, then $F_i(\psi) = F_i(\psi') \det^2(\alpha) \det^2(\gamma) \det^2(\delta)$, $i = 3, 4$.

In Eq. (2), let $\gamma = I$, then $F_i(\psi) = F_i(\psi') \det^2(\alpha) \det^2(\beta) \det^2(\delta)$, $i = 5, 6$.

In Eq. (2), let $\delta = I$, then $F_i(\psi) = F_i(\psi') \det^2(\alpha) \det^2(\beta) \det^2(\gamma)$, $i = 7, 8$.

In Eq. (2), let $\alpha = I$ and $\beta = I$, i.e. $|\psi\rangle = I \otimes I \otimes \gamma \otimes \delta |\psi'\rangle$, then $F_i(\psi) = F_i(\psi') \det^2(\gamma) \det^2(\delta)$, $i = 9, 10$.

Next let $|\psi\rangle$ be SLOCC equivalent to $|\psi'\rangle$ in Eq. (2). By solving the matrix equation in Eq. (2), we obtain amplitudes $a_i$ of state $|\psi\rangle$ in Eq. (2). Then we can calculate $F_i$ of state $|\psi\rangle$, i.e., the $F_i$ of class $|\psi'\rangle$. We compute $F_i$ of all the degenerated entanglement classes and 28 true entanglement classes. See Tables 2.1, 2.2 and 5 and Appendix A. If $F_i$ do not vanish for some classes, then we give the expressions of $F_i$ in Appendix A. We list the properties of $F_i$ of the 28 true entanglement classes in Tables 2.1 and 2.2 and of $F_i$ of all the degenerated entanglement classes in Table 5. For the derivations of the properties of $F_i$, see Appendix A. We also compute all the $F_i$ of the 28 true entanglement states, see Tables 3.1. and 3.2.

## 2.3 Semi-invariants $D_1$, $D_2$ and $D_3$

In [15], we computed the following expressions of $D_1$, $D_2$ and $D_3$ for $|GHZ\rangle$, $|W\rangle$. Using the $D_1$, $D_2$ and $D_3$, we found a true entanglement state $|C_4\rangle$ which is distinct from $|GHZ\rangle$, $|W\rangle$ and $|\phi_4\rangle$[6]. We define $D_i(\psi)$ for state $\psi$ as follows.

$$D_1(\psi) = (a_1a_4 - a_0a_5)(a_{11}a_{14} - a_{10}a_{15}) - (a_3a_6 - a_2a_7)(a_9a_{12} - a_8a_{13}), \quad (9)$$

$$D_2(\psi) = (a_4a_7 - a_5a_6)(a_8a_{11} - a_9a_{10}) - (a_0a_3 - a_1a_2)(a_{12}a_{15} - a_{13}a_{14}), \quad (10)$$

$$D_3(\psi) = (a_3a_5 - a_1a_7)(a_{10}a_{12} - a_8a_{14}) - (a_2a_4 - a_0a_6)(a_{11}a_{13} - a_9a_{15}). \quad (11)$$

Then by induction we can demonstrate that $D_i(\psi)$ have the following interesting properties, which are also called as semi-invariants. The properties can also be verified by substituting the amplitudes $a_i$ of state $|\psi\rangle$ in Eq. (2) into $D_i(\psi)$ $i = 1, 2$ and 3. For state $|\psi'\rangle$, let $D_i(\psi')$ be be obtained from $D_i(\psi)$ by replacing $a$ in $D_i(\psi)$ by $b$. Then

In Eq. (2), let $\alpha = I$ and $\gamma = I$, i.e. $|\psi\rangle = I \otimes \beta \otimes I \otimes \delta|\psi'\rangle$, then $D_1(\psi) = D_1(\psi') \det^2(\beta) \det^2(\delta)$.

In Eq. (2), let $\alpha = I$ and $\beta = I$, then $D_2(\psi) = D_2(\psi') \det^2(\gamma) \det^2(\delta)$.

In Eq. (2), let $\alpha = I$ and $\delta = I$, then $D_3(\psi) = D_3(\psi') \det^2(\beta) \det^2(\gamma)$.

Next let $|\psi\rangle$ be SLOCC equivalent to $|\psi'\rangle$ in Eq. (2). By solving the matrix equation in Eq. (2), we obtain amplitudes $a_i$ of state $|\psi\rangle$ in Eq. (2). Then we can calculate $D_i$ of state $|\psi\rangle$, i.e., the $D_i$ of class $|\psi'\rangle$. We compute the values of $D_1, D_2$ and $D_3$ of the degenerated SLOCC equivalence classes (see Table 4). We give the values of $D_1, D_2$ and $D_3$ of the 28 true entanglement classes in Tables 1.1 and 1.2 and of the 28 true entanglement states in Tables 3.1 and 3.2. We also calculate the values of $D_1, D_2$ and $D_3$ of states $|GHZ\rangle_{12} \otimes |GHZ\rangle_{34}$, $|GHZ\rangle_{13} \otimes |GHZ\rangle_{24}$ and $|GHZ\rangle_{14} \otimes |GHZ\rangle_{23}$, see Table 6. If $D_i = 0$ for some $i$ and for some class in Tables 1.1 and 1.2 and 4, then it implies that $D_i = 0$ for the $i$ and for all the states of the class. If $D_i$ is "opt" for some $i$ and for some class in Tables 1.1 and 1.2 and 4, then it means that $D_i = 0$ for the $i$ and for some states of the class while for other states of the class $D_i \neq 0$. If $D_i$ is "opt", then we give the expression of $D_i$ in Appendix A. For example, for class $|\kappa_4\rangle$ in Table 1.1, $D_1$ is "opt", $D_2$ is "opt" and $D_3 = 0$. It says that for some state of class $|\kappa_4\rangle$ in Table 1.1, $D_1 \neq 0$ and $D_2 \neq 0$ but $D_3 = 0$ for every state of class $|\kappa_4\rangle$. However, for state $|\kappa_4\rangle$ in Table 3.1, $D_i = 0$, where $i = 1, 2$ and 3,

# 3 The invariant, semi-invariants for SLOCC classification

## 3.1 The representatives of true entanglement classes

It is well known that states $|GHZ\rangle$, $|W\rangle$, $|\phi_4\rangle$ and $|C_4\rangle$ are the representatives of disjoint true entanglement classes of four-qubits. Utilizing the SLOCC invariant $IV$ and the semi-invariants $F_i$ and $D_i$ of four-qubits, we find 28 distinct true entanglement classes. The representatives of the classes are listed below.

1. From the construction of $|\phi_4\rangle$, we do the following computation tests. From all the 15 true entanglement states: $(|0\rangle + |i\rangle + |j\rangle - |15\rangle)/2$, where $|i\rangle, |j\rangle \in \{|3\rangle, |5\rangle, |6\rangle, |9\rangle, |10\rangle, |12\rangle\}$ which is obtained from $|C_4\rangle$, we find the representatives of 7 different true entanglement classes. They are $|GHZ\rangle, |\phi_4\rangle, |\psi_4\rangle$, $|\mu_4\rangle$, $|\kappa_4\rangle$, $|E_4\rangle$ and $|L_4\rangle$.

2. From [9], we consider the states of the following forms: $(|0\rangle + |i\rangle + |j\rangle + |k\rangle + |l\rangle + |15\rangle)/\sqrt{6}$, where $|i\rangle, |j\rangle, |k\rangle, |l\rangle \in \{|3\rangle, |5\rangle, |6\rangle, |9\rangle, |10\rangle, |12\rangle$. There are 15 true entanglement states, of which seven are chosen as the representatives of different true entanglement classes. They are $|C_4\rangle, |\pi_4\rangle, |\sigma_4\rangle, |\rho_4\rangle, |\xi_4\rangle, |\epsilon_4\rangle$ and $|\theta_4\rangle$.

3. From the 15 true entanglement states: $(|0\rangle+|i\rangle+|j\rangle+|k\rangle+|l\rangle-|15\rangle)/\sqrt{6}$, where $|i\rangle, |j\rangle, |k\rangle, |l\rangle \in \{|3\rangle, |5\rangle, |6\rangle, |9\rangle, |10\rangle, |12\rangle$, we choose $|\chi_4\rangle$ as a representative.

4. Consider the true entanglement states of the following forms: $|3\rangle + |x\rangle + |12\rangle$, where $|x\rangle \in \{|5\rangle, |6\rangle, |9\rangle, |10\rangle\}$; $|5\rangle+|x\rangle+|10\rangle$, where $|x\rangle \in \{|3\rangle, |6\rangle, |9\rangle, |12\rangle\}$; $|6\rangle + |x\rangle + |9\rangle$, where $|x\rangle \in \{|3\rangle, |5\rangle, |10\rangle, |12\rangle\}$. Notice that $|3\rangle$ and $|12\rangle$, $|5\rangle$ and $|10\rangle|$ and $6\rangle$ and $|9\rangle$ are dual, respectively. From the 12 true entanglement states, we find three inequivalent true entanglement states. They are $|H_4\rangle, |\lambda_4\rangle$ and $|M_4\rangle$.

5. The classes whose representatives have 3 product terms

Let $S_1 = (001)^T$, $S_2 = (010)^T$, $S_3 = (100)^T$, $V_1 = (011)^T$, $V_2 = (101)^T$, $V_3 = (110)^T$. Consider the permutations: $S_iS_jV_iV_j$. For example, $S_1S_2V_1V_2$ is considered as a matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$. Each row of the matrix is considered a basic state of four-qubits. Thus, the matrix can be considered a state $(|1\rangle + |6\rangle + |11\rangle)/\sqrt{3}$. From the permutations, we find representatives: $|\varphi_4\rangle$, $|\tau_4\rangle$, $|\vartheta_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\varsigma_4\rangle$.

6. $|\omega_4\rangle$ is $L_{0_{5\oplus 3}}$ in [9]. From states of the forms $(|x\rangle + |5\rangle + |y\rangle + |z\rangle)/2$, where $x + 5 + y + z = 27$, we choose $|\upsilon_4\rangle$, $|\varpi_4\rangle$ and $|\omega_4\rangle$ as representatives.

7. Up to permutations of the qubits, each one of the following groups is considered as one true entanglement class. However, we don't show that different groups can not be obtained up to permutations of the qubits.

We list the 28 true entanglement classes as follows.

1. $|GHZ\rangle = (|0\rangle + |15\rangle)/\sqrt{2}$,

2. $|W\rangle = (|1\rangle + |2\rangle + |4\rangle + |8\rangle)/2$,

3. $|C_4\rangle = (|3\rangle + |5\rangle + |6\rangle + |9\rangle + |10\rangle + |12\rangle)/\sqrt{6}$,

4.

$|\kappa_4\rangle = (|0\rangle + |3\rangle + |10\rangle - |15\rangle)/2$,

$|E_4\rangle = (|0\rangle + |5\rangle + |9\rangle - |15\rangle)/2$,

$|L_4\rangle = (|0\rangle + |3\rangle + |9\rangle - |15\rangle)/2$,

5.

$|H_4\rangle = (|3\rangle + |6\rangle + |12\rangle)/\sqrt{3}$,

$|\lambda_4\rangle = (|5\rangle + |6\rangle + |10\rangle)/\sqrt{3}$,

$|M_4\rangle = (|3\rangle + |5\rangle + |12\rangle)/\sqrt{3}$,

6.
$|\pi_4\rangle = (|0\rangle + |3\rangle + |5\rangle + |6\rangle + |10\rangle + |15\rangle)/\sqrt{6},$
$|\theta_4\rangle = (|0\rangle + |5\rangle + |6\rangle + |10\rangle + |12\rangle + |15\rangle)/\sqrt{6},$
$|\sigma_4\rangle = (|0\rangle + |3\rangle + |9\rangle + |10\rangle + |12\rangle + |15\rangle)/\sqrt{6},$
$|\rho_4\rangle = (|0\rangle + |3\rangle + |6\rangle + |10\rangle + |12\rangle + |15\rangle)/\sqrt{6},$
$|\xi_4\rangle = (|0\rangle + |6\rangle + |9\rangle + |10\rangle + |12\rangle + |15\rangle)/\sqrt{6},$
$|\epsilon_4\rangle = (|0\rangle + |3\rangle + |6\rangle + |9\rangle + |10\rangle + |15\rangle)/\sqrt{6},$
7.
$|\chi_4\rangle = (|0\rangle + |3\rangle + |6\rangle + |10\rangle + |12\rangle - |15\rangle)/\sqrt{6},$
8.
$|\psi_4\rangle = (|0\rangle + |5\rangle + |10\rangle - |15\rangle)/2,$
$|\phi_4\rangle = (|0\rangle + |3\rangle + |12\rangle - |15\rangle)/2,$
$|\mu_4\rangle = (|0\rangle + |6\rangle + |9\rangle - |15\rangle)/2,$
9.
$|\varphi_4\rangle = (|1\rangle + |6\rangle + |11\rangle)/\sqrt{3},$
$|\vartheta_4\rangle = (|2\rangle + |5\rangle + |11\rangle)/\sqrt{3},$
$|\tau_4\rangle = (|1\rangle + |7\rangle + |10\rangle)/\sqrt{3},$
$|\varrho_4\rangle = (|2\rangle + |7\rangle + |9\rangle)/\sqrt{3},$
10.
$|\zeta_4\rangle = (|0\rangle + |11\rangle + |12\rangle)/\sqrt{3},$
$|\iota_4\rangle = (|0\rangle + |3\rangle + |13\rangle)/\sqrt{3},$
11.
$|\upsilon_4\rangle = (|2\rangle + |5\rangle + |9\rangle + |11\rangle/2,$
12.
$|\omega_4\rangle = (|0\rangle + |5\rangle + |8\rangle + |14\rangle)/2,$
13.
$|\varpi_4\rangle = (|2\rangle + |5\rangle + |8\rangle + |12\rangle)/2$

### 3.2 The sufficient conditions for a true entanglement state

From Tables 1, 2, 4 and 5, it is not difficult to see that a state is a true entanglement state if the state satisfies one of the following conditions.
(1). $IV = 0$ and $D_i \neq 0$, where $i = 1, 2$ or 3,
(2). $IV \neq 0$ and $F_i \neq 0$, where $i = 1, 2, 3, 4, 5, 6, 7$ or 8,
(3). $IV \neq 0$ and $D_i \neq 0$ and $D_j \neq 0$.

## 4 At least 28 distinct true entanglement classes

### 4.1 Degenerated entanglement classes

The authors in [17] gave an upper bound for the number of degenerate $(N+1)$-entanglement classes in terms of the number of $N$-partite entanglement classes. In this paper, we give an exact recursive formula for the number of degenerate entanglement classes of $n$-qubits, see Appendix B. By the recursive formula, for five-qubits, there are $5 * t(4) + 66$ distinct degenerated SLOCC entanglement

classes, where $t(4)$ is the number of the true SLOCC entanglement classes for four-qubits. We only use combinatory analysis to derive the recursive formula. The authors in [18] declared there are 16 true entanglement SLOCC classes for four-qubits and at most 170 degenerate entanglement classes for five-qubits. If so, by our recursive formula there would be 146 degenerated SLOCC entanglement classes for five-qubits. However, in this paper, we report there are at least 28 true entanglement SLOCC classes for four-qubits. Thus, by the recursive formula there are at least 206 degenerated SLOCC classes for five-qubits. From the recursive formula, we know that the most of the degenerated entanglement classes of $n$-qubits are the $(n-1)$-qubit true entanglement with a separate qubit like $A-(BC....Z)_{n-1}$, where $(BC....Z)_{n-1}$ is truly entangled.

For four-qubits, by computing, we obtain the SLOCC invariant, the semi-invariants $F_i$ and $D_i$ of all the degenerated entanglement classes. See Tables 4, 5 and 6. For example, value $F$ of three-qubit $GHZ$ entanglement accompanied with a separable qubit does not vanish. This proof is given as follows. Other proofs are omitted.

By the definition of $F$ and section 3.1 of [15], it is easy to obtain this result.

For class $|GHZ\rangle_{ABC}\otimes(s|0\rangle+t|1\rangle)_D$, by section 3.1 of [15], $(a_0a_{14}-a_4a_{10}+a_2a_{12}-a_6a_8)^2 \neq 4(a_4a_8-a_0a_{12})(a_6a_{10}-a_2a_{14})$ or

$(a_1a_{15}-a_5a_{11}+a_3a_{13}-a_7a_9)^2 \neq 4(a_5a_9-a_1a_{13})(a_7a_{11}-a_3a_{15})$, and other $F_i$ vanish.

For class $|GHZ\rangle_{ABD}\otimes(s|0\rangle+t|1\rangle)_C$,

$(a_0a_{13}-a_4a_9+a_1a_{12}-a_5a_8)^2 \neq 4(a_4a_8-a_0a_{12})(a_5a_9-a_1a_{13})$

or $(a_2a_{15}-a_6a_{11}+a_3a_{14}-a_7a_{10})^2 \neq 4(a_6a_{10}-a_2a_{14})(a_7a_{11}-a_3a_{15})$,

and other $F_i$ vanish.

For class $|GHZ\rangle_{ACD}\otimes(s|0\rangle+t|1\rangle)_B$ ,

$(a_0a_{11}-a_2a_9+a_1a_{10}-a_3a_8)^2 \neq 4(a_2a_8-a_0a_{10})(a_3a_9-a_1a_{11})$

or $(a_4a_{15}-a_6a_{13}+a_5a_{14}-a_7a_{12})^2 \neq 4(a_6a_{12}-a_4a_{14})(a_7a_{13}-a_5a_{15})$.

For class $(s|0\rangle+t|1\rangle)_A\otimes|GHZ\rangle_{BCD}$,

$(a_0a_7-a_2a_5+(a_1a_6-a_3a_4))^2 \neq 4(a_2a_4-a_0a_6)(a_3a_5-a_1a_7)$

or $((a_8a_{15}-a_{11}a_{12})+(a_9a_{14}-a_{10}a_{13}))^2 \neq 4(a_{11}a_{13}-a_9a_{15})(a_{10}a_{12}-a_8a_{14})$.

### 4.2 The 28 classes in Tables 1.1 and 1.2 are true entanglement classes.

It is known that classes $|GHZ\rangle$, $|W\rangle$, $|\phi_4\rangle$[6] and $|C_4\rangle$ [15] are inequivalent true entanglement classes.

Part 1. The classes in Table 1.1 are true entanglement classes.

Since for the classes in Table 1.1, $IV\neq 0$ and $F>0$, so we only need to show that the classes in Table 1.1 are distinct from the degenerated entanglement classes $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$. However, $F_i$ in Tables 3.1 do not satisfy the properties of $F_i$ of classes $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ or $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$ in Table 5. Hence, the classes in Table 1.1 are true entanglement classes.

Part 2. The classes in Table 1.2 are true entanglement classes.

Since $IV=0$ for classes in Table 1.2, the classes in Table 1.2 are distinct from the degenerated entanglement classes $|GHZ\rangle_{12}\otimes|GHZ\rangle_{34}$, $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$. For some states of classes $|\chi_4\rangle$, $|\upsilon_4\rangle$, $|\varpi_4\rangle$, $|\psi_4\rangle$, $|\phi_4\rangle$, $|\mu_4\rangle$, $|\varphi_4\rangle$, $|\varsigma_4\rangle$ and $|\vartheta_4\rangle$, always $D_i\neq 0$ for some $i$, see Appendix A. However, all the classes in Table 4 except for $|GHZ\rangle_{12}\otimes|GHZ\rangle_{34}$, $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$, require $D_i=0$, where $i=1,2$, and 3, see Table 4. So classes $|\chi_4\rangle$, $|\upsilon_4\rangle$, $|\varpi_4\rangle$, $|\psi_4\rangle$, $|\phi_4\rangle$, $|\mu_4\rangle$, $|\varphi_4\rangle$, $|\varsigma_4\rangle$ and $|\vartheta_4\rangle$ are not degenerated entanglement classes. The classes $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\omega_4\rangle$ in Table 1.2 are not degenerated entanglement classes because the properties of $F_i$ of classes $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\omega_4\rangle$ in Table 2.2 do not satisfy the conditions of $F_i$ in Table 5.

### 4.3 The 28 classes in Tables 1.1 and 1.2 are distinct each other.

Clearly, the classes in Table 1.1 are distinct from the ones in Table 1.2 because the values of $IV$ of all the classes in Table 1.2 are zero while the values of $IV$ of all the classes in Table 1.1 are not zero.

Part 1. Let us show that the classes in Table 1.1 are distinct each other.

For class $|GHZ\rangle$, $D_i=0$, where $i=1,2$ and 3, see Table 1.1. However, always $D_i\neq 0$ for some $i$ and for some states of other classes in Table 1.1. Consequently, class $|GHZ\rangle$ is distinct from other classes in Table 1.1. For state $|C_4\rangle$, $D_i\neq 0$, where $i=1,2$ and 3, see Table 3.1. It is easy to see from Table 1.1 that for the other classes, always $D_i=0$ for some $i$. For example, $D_3=0$ for class $|\kappa_4\rangle$, see Table 1.1. It implies that $D_3=0$ for every state of class $|\kappa_4\rangle$. Therefore, class $|C_4\rangle$ is different from other classes in Table 1.1.

For some states of class $|\kappa_4\rangle$, $D_1\neq 0$ and $D_2\neq 0$, see the case for class $|\kappa_4\rangle$ in Appendix A, and for every state of class $|\kappa_4\rangle$, $D_3=0$. Therefore class $|\kappa_4\rangle$ is different from the last 11 classes in Table 1.1. As well, classes $|E_4\rangle$ and $|L_4\rangle$ are different each other and from the last 9 classes in Table 1.1.

Let us demonstrate that class $|H_4\rangle$ is different from classes $|\pi_4\rangle$ and $|\theta_4\rangle$. For state $|H_4\rangle$, $F_9\neq 0$ and $F_i=0$ when $i\neq 9$, see Table 3.1. Thus, state $|H_4\rangle$ does not satisfy the conditions of $F_i$ of classes $|\pi_4\rangle$ or $|\theta_4\rangle$, see Table 2.1. Therefore, class $|H_4\rangle$ is different from classes $|\pi_4\rangle$ and $|\theta_4\rangle$. For some states of class $|H_4\rangle$, $D_1\neq 0$ and for every state of class $|H_4\rangle$, $D_2=0$ and $D_3=0$, therefore class $|H_4\rangle$ is different from $|\lambda_4\rangle$ and $|M_4\rangle$ and the last 4 classes in Table 1.1. As well, $|\lambda_4\rangle$ and $|M_4\rangle$ are different each other and from the last 6 classes in Table 1.1.

From Table 2.1, it is easy to see that for the last six classes: $|\pi_4\rangle$, $|\theta_4\rangle$, $|\sigma_4\rangle$, $|\rho_4\rangle$, $|\xi_4\rangle$ and $|\epsilon_4\rangle$, the properties of $F_i$ are disjoint, hence they are distinct each other.

Part 2. We argue that the classes in Table 1.2 are distinct each other.

Since $F=0$ for class $|W\rangle$ and $F\neq 0$ for other classes in Table 1.2, class $|W\rangle$ is distinct from other classes in Table 1.2. For some states of class $|\chi_4\rangle$, $D_i\neq 0$, where $i=1,2$ and 3, see the cases for class $|\chi_4\rangle$ in Appendix A. As discussed for class $|C_4\rangle$ in Part 1, class $|\chi_4\rangle$ is distinct from other classes in Table 1.2.

For some states of class $|\upsilon_4\rangle$, $D_2\neq 0$ and $D_3\neq 0$, see the cases for class $|\upsilon_4\rangle$ in Appendix A, and for every state of class $|\upsilon_4\rangle$, $D_1=0$, see Table 1.2.

Therefore, class $|\upsilon_4\rangle$ is different from other classes in Table 1.2. We omit the similar discussions which can be found in Part 1. We need to argue that $|\psi_4\rangle$ and $|\varphi_4\rangle$ are different each other. For state $|\psi_4\rangle$, $F_1 = 0$ and $F_2 = 0$, see Table 3.2. This conflicts that $|F_1| + |F_2| \neq 0$ for class $|\varphi_4\rangle$, see Table 2.2. This is done. As well, $|\phi_4\rangle$ and $|\varsigma_4\rangle$ are different each other, so are $|\mu_4\rangle$ and $|\vartheta_4\rangle$. What remains is to explain that $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$ and $|\omega_4\rangle$ are distinct each other. This is obvious because the properties of their $F_i$ are disjoint, see Table 2.2.

Conjecture:

There should be many many true SLOCC entanglement classes. We can show $\frac{1}{\sqrt{6}}(\sqrt{2}|15\rangle + |8\rangle + |4\rangle + |2\rangle + |1\rangle)$ in [22] is a true entanglement state which does not belong to the 28 classes because the state has the following properties.

$IV = 0$, $D_1$: opt, $D_2$: opt, $D_3$: opt, $|F_1| + |F_2| \neq 0, |F_3| + |F_4| \neq 0, |F_5| + |F_6| \neq 0, |F_7| + |F_8| \neq 0.$

Table 1.1.

The SLOCC invariants of true entanglement classes

| classes | $F$ | $D_1$ | $D_2$ | $D_3$ | $IV$ |
|---|---|---|---|---|---|
| $\lvert GHZ\rangle$ | $>0$ | $=0$ | $=0$ | $=0$ | $\neq 0$ |
| $\lvert C_4\rangle$ | $>0$ | opt | opt | opt | $\neq 0$ |
| $\lvert \kappa_4\rangle$ | $>0$ | opt | opt | $=0$ | $\neq 0$ |
| $\lvert E_4\rangle$ | $>0$ | opt | $=0$ | opt | $\neq 0$ |
| $\lvert L_4\rangle$ | $>0$ | $=0$ | opt | opt | $\neq 0$ |
| $\lvert H_4\rangle$ | $>0$ | opt | $=0$ | $=0$ | $\neq 0$ |
| $\lvert \lambda_4\rangle$ | $>0$ | $=0$ | opt | $=0$ | $\neq 0$ |
| $\lvert M_4\rangle$ | $>0$ | $=0$ | $=0$ | opt | $\neq 0$ |
| $\lvert \pi_4\rangle$ | $>0$ | opt | $=0$ | $=0$ | $\neq 0$ |
| $\lvert \theta_4\rangle$ | $>0$ | opt | $=0$ | $=0$ | $\neq 0$ |
| $\lvert \sigma_4\rangle$ | $>0$ | $=0$ | opt | $=0$ | $\neq 0$ |
| $\lvert \rho_4\rangle$ | $>0$ | $=0$ | opt | $=0$ | $\neq 0$ |
| $\lvert \xi_4\rangle$ | $>0$ | $=0$ | $=0$ | opt | $\neq 0$ |
| $\lvert \epsilon_4\rangle$ | $>0$ | $=0$ | $=0$ | opt | $\neq 0$ |

"opt" means that $D_i$ of some states of the class are zero while $D_i$ of other states are not zero.

Table 1.2 The SLOCC invariants of true entanglement classes

| classes | $F$ | $D_1$ | $D_2$ | $D_3$ | $IV$ |
|---|---|---|---|---|---|
| $\|W\rangle$ | $=0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| $\|\chi_4\rangle$ | $>0$ | opt | opt | opt | $=0$ |
| $\|\upsilon_4\rangle$ | $>0$ | $=0$ | opt | opt | $=0$ |
| $\|\varpi_4\rangle$ | $>0$ | opt | $=0$ | opt | $=0$ |
| $\|\psi_4\rangle$ | $>0$ | opt | $=0$ | $=0$ | $=0$ |
| $\|\phi_4\rangle$ | $>0$ | $=0$ | opt | $=0$ | $=0$ |
| $\|\mu_4\rangle$ | $>0$ | $=0$ | $=0$ | opt | $=0$ |
| $\|\varphi_4\rangle$ | $>0$ | opt | $=0$ | $=0$ | $=0$ |
| $\|\zeta_4\rangle$ | $>0$ | $=0$ | opt | $=0$ | $=0$ |
| $\|\vartheta_4\rangle$ | $>0$ | $=0$ | $=0$ | opt | $=0$ |
| $\|\tau_4\rangle$ | $>0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| $\|\varrho_4\rangle$ | $>0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| $\|\iota_4\rangle$ | $>0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| $\|\omega_4\rangle$ | $>0$ | $=0$ | $=0$ | $=0$ | $=0$ |

Table 2.1.

The properties of $F_i$ for true entanglement classes

| classes | $F_i$ | $F_i = 0, i =$ |
|---|---|---|
| $\|GHZ\rangle$ | #0 | |
| $\|C_4\rangle$ | #4 | |
| $\|\kappa_4\rangle, \|E_4\rangle, \|L_4\rangle, \|H_4\rangle, \|\lambda_4\rangle, \|M_4\rangle$ | #0 | |
| $\|\pi_4\rangle$ | $\|F_1\| + \|F_2\| \neq 0, \|F_5\| + \|F_6\| \neq 0,$ #2 | $3, 4, 7, 8$ |
| $\|\theta_4\rangle$ | $\|F_3\| + \|F_4\| \neq 0, \|F_7\| + \|F_8\| \neq 0,$ #3 | $1, 2, 5, 6$ |
| $\|\sigma_4\rangle$ | $\|F_1\| + \|F_2\| \neq 0, \|F_3\| + \|F_4\| \neq 0,$ #1 | $5, 6, 7, 8$ |
| $\|\rho_4\rangle$ | $\|F_5\| + \|F_6\| \neq 0, \|F_7\| + \|F_8\| \neq 0$ | $1, 2, 3, 4, 9, 10$ |
| $\|\xi_4\rangle$ | $\|F_1\| + \|F_2\| \neq 0, \|F_7\| + \|F_8\| \neq 0,$ #2 | $3, 4, 5, 6$ |
| $\|\epsilon_4\rangle$ | $\|F_3\| + \|F_4\| \neq 0, \|F_5\| + \|F_6\| \neq 0,$ #3 | $1, 2, 7, 8$ |

#0 : If $F_1 F_2 = 0$ and $F_3 F_4 = 0$ then $F_9 = 0$ and $F_{10} \neq 0$ or $F_9 \neq 0$ and $F_{10} = 0$.

#1 : If $F_1 F_2 = 0$ and $F_3 F_4 = 0$ then $F_9 = F_{10} = 0$.

#2 : If $F_1 F_2 = 0$ then $F_9 = F_{10} \neq 0$.

#3 : If $F_3 F_4 = 0$ then $F_9 = F_{10} \neq 0$.

#4 : If $F_i = F_j = F_k = 0$, where $1 \leq i < j < k \leq 4$, then $|F_9| + |F_{10}| \neq 0$ and $F_9 F_{10} = 0$.

Table 2.2 The properties of $F_i$ for true entanglement classes

| classes | $F_i$ | $F_i = 0, i =$ |
|---|---|---|
| $\lvert W\rangle$ | | all $i$ |
| $\lvert\chi_4\rangle$ | $\lvert F_5\rvert + \lvert F_6\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0$,#0 | |
| $\lvert\upsilon_4\rangle$ | $\lvert F_1\rvert + \lvert F_2\rvert \neq 0, \lvert F_3\rvert + \lvert F_4\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0$, #1 | $i = 5, 6$ |
| $\lvert\varpi_4\rangle$ | $\lvert F_1\rvert + \lvert F_2\rvert \neq 0, \lvert F_5\rvert + \lvert F_6\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0, F_9 = F_{10}, F_1F_2 = (F_9)^2$ | $3, 4$ |
| $\lvert\psi_4\rangle$ | $F_9 = F_{10}$, #5 | |
| $\lvert\phi_4\rangle$ | #0 | |
| $\lvert\mu_4\rangle$ | $F_9 = F_{10}$, #5 | |
| $\lvert\varphi_4\rangle$ | $\lvert F_1\rvert + \lvert F_2\rvert \neq 0, \lvert F_5\rvert + \lvert F_6\rvert \neq 0, F_9 = F_{10}, F_1F_2 = (F_9)^2$ | $3, 4, 7, 8$ |
| $\lvert\zeta_4\rangle$ | $\lvert F_1\rvert + \lvert F_2\rvert \neq 0, \lvert F_3\rvert + \lvert F_4\rvert \neq 0$, #1 | $5, 6, 7, 8$ |
| $\lvert\vartheta_4\rangle$ | $\lvert F_1\rvert + \lvert F_2\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0, F_9 = F_{10}, F_1F_2 = (F_9)^2$ | $3, 4, 5, 6$ |
| $\lvert\tau_4\rangle$ | $\lvert F_3\rvert + \lvert F_4\rvert \neq 0, \lvert F_5\rvert + \lvert F_6\rvert \neq 0, F_9 = F_{10}, F_3F_4 = (F_9)^2$ | $1, 2, 7, 8$ |
| $\lvert\varrho_4\rangle$ | $\lvert F_3\rvert + \lvert F_4\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0, F_9 = F_{10}, F_3F_4 = (F_9)^2$ | $1, 2, 5, 6$ |
| $\lvert\iota_4\rangle$ | $\lvert F_5\rvert + \lvert F_6\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0,$ | $1, 2, 3, 4, 9, 10$ |
| $\lvert\omega_4\rangle$ | $\lvert F_3\rvert + \lvert F_4\rvert \neq 0, \lvert F_5\rvert + \lvert F_6\rvert \neq 0, \lvert F_7\rvert + \lvert F_8\rvert \neq 0, F_9 = F_{10}, F_3F_4 = (F_9)^2$ | $i = 1, 2$ |

#5 : If $F_i = F_j = F_k = 0$, where $1 \leq i < j < k \leq 4$, then $F_9 \neq 0$.

Table 3.1. The properties of $D_i$ and $F_i$ for the true entanglement states

| States | $D_1$ | $D_2$ | $D_3$ | $F_i \neq 0$, when $i =$ |
|---|---|---|---|---|
| $\lvert GHZ\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert C_4\rangle$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | 9 |
| $\lvert\kappa_4\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert E_4\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert L_4\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert H_4\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert\lambda_4\rangle$ | $= 0$ | $\neq 0$ | $= 0$ | 10 |
| $\lvert M_4\rangle$ | $= 0$ | $= 0$ | $= 0$ | 9 |
| $\lvert\pi_4\rangle$ | $\neq 0$ | $= 0$ | $= 0$ | $1, 6, 9, 10,$ |
| $\lvert\theta_4\rangle$ | $\neq 0$ | $= 0$ | $= 0$ | $4, 7, 9, 10$ |
| $\lvert\sigma_4\rangle$ | $= 0$ | $\neq 0$ | $= 0$ | $2, 3$ |
| $\lvert\rho_4\rangle$ | $= 0$ | $\neq 0$ | $= 0$ | $6, 7$ |
| $\lvert\xi_4\rangle$ | $= 0$ | $= 0$ | $\neq 0$ | $2, 7, 9, 10,$ |
| $\lvert\epsilon_4\rangle$ | $= 0$ | $= 0$ | $\neq 0$ | $3, 6, 9, 10,$ |

Table 3.2 The properties of $D_i$ and $F_i$ for the true entanglement states

| States | $D_1$ | $D_2$ | $D_3$ | $F_i \neq 0$, when $i =$ |
|---|---|---|---|---|
| $\|W\rangle$ | $=0$ | $=0$ | $=0$ | |
| $\|\chi_4\rangle$ | $=0$ | $\neq 0$ | $=0$ | $6,7,9$ |
| $\|\upsilon_4\rangle$ | $=0$ | $=0$ | $=0$ | $1,3,8$ |
| $\|\varpi_4\rangle$ | $=0$ | $=0$ | $=0$ | $1,5,7$ |
| $\|\psi_4\rangle$ | $\neq 0$ | $=0$ | $=0$ | $3,4,9,10$ |
| $\|\phi_4\rangle$ | $=0$ | $\neq 0$ | $=0$ | $9$ |
| $\|\mu_4\rangle$ | $=0$ | $=0$ | $\neq 0$ | $9,10$ |
| $\|\varphi_4\rangle$ | $=0$ | $=0$ | $=0$ | $1,6$ |
| $\|\zeta_4\rangle$ | $=0$ | $=0$ | $=0$ | $2,3$ |
| $\|\vartheta_4\rangle$ | $=0$ | $=0$ | $=0$ | $1,8$ |
| $\|\tau_4\rangle$ | $=0$ | $=0$ | $=0$ | $3,6$ |
| $\|\varrho_4\rangle$ | $=0$ | $=0$ | $=0$ | $3,8$ |
| $\|\iota_4\rangle$ | $=0$ | $=0$ | $=0$ | $5,8$ |
| $\|\omega_4\rangle$ | $=0$ | $=0$ | $=0$ | $4,5,7,$ |

Table 4. The invariants of the degenerated entanglement classes

| Classes | $IV$ | $D_1$ | $D_2$ | $D_3$ | $F$ |
|---|---|---|---|---|---|
| $\|GHZ\rangle_{123} \otimes (s\|0\rangle + t\|1\rangle)_4$ | $=0$ | $=0$ | $=0$ | $=0$ | $>0$ |
| $\|GHZ\rangle_{124} \otimes (s\|0\rangle + t\|1\rangle)_3$ | $=0$ | $=0$ | $=0$ | $=0$ | $>0$ |
| $\|GHZ\rangle_{134} \otimes (s\|0\rangle + t\|1\rangle)_2$ | $=0$ | $=0$ | $=0$ | $=0$ | $>0$ |
| $(s\|0\rangle + t\|1\rangle)_1 \otimes \|GHZ\rangle_{234}$ | $=0$ | $=0$ | $=0$ | $=0$ | $>0$ |
| $\|W\rangle \otimes (s\|0\rangle + t\|1\rangle)_i$ | $=0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| $\|GHZ\rangle_{12} \otimes \|GHZ\rangle_{34}$ | $\neq 0$ | $=0$ | opt | $=0$ | $=0$ |
| $\|GHZ\rangle_{13} \otimes \|GHZ\rangle_{24}$ | $\neq 0$ | opt | $=0$ | $=0$ | $>0$ |
| $\|GHZ\rangle_{14} \otimes \|GHZ\rangle_{23}$ | $\neq 0$ | $=0$ | $=0$ | opt | $>0$ |
| only two qubits are entangled | $=0$ | $=0$ | $=0$ | $=0$ | $=0$ |
| separate states | $=0$ | $=0$ | $=0$ | $=0$ | $=0$ |

In $|W\rangle \otimes (s|0\rangle + t|1\rangle)_i$, $i = 1,2,3,4$.

Table 5. The properties of $F_i$ for the degenerated entanglement classes

| Classes | $F_i = 0$ | |
|---|---|---|
| $\|GHZ\rangle_{123} \otimes (s\|0\rangle + t\|1\rangle)_4$ | $i \neq 7,8$ | $\|F_7\| + \|F_8\| \neq 0$ |
| $\|GHZ\rangle_{124} \otimes (s\|0\rangle + t\|1\rangle)_3$ | $i \neq 5,6$ | $\|F_5\| + \|F_6\| \neq 0$ |
| $\|GHZ\rangle_{134} \otimes (s\|0\rangle + t\|1\rangle)_2$ | $i \neq 3,4,9,10$ | $\|F_3\| + \|F_4\| \neq 0, F_9 = F_{10}, F_3F_4 = (F_9)^2$ |
| $(s\|0\rangle + t\|1\rangle)_1 \otimes \|GHZ\rangle_{234}$ | $i \neq 1,2,9,10$ | $\|F_1\| + \|F_2\| \neq 0, F_9 = F_{10}, F_1F_2 = (F_9)^2$ |
| $\|W\rangle \otimes (s\|0\rangle + t\|1\rangle)_i$ | All $F_i = 0$ | |
| $\|GHZ\rangle_{12} \otimes \|GHZ\rangle_{34}$ | All $F_i = 0$ | |
| $\|GHZ\rangle_{13} \otimes \|GHZ\rangle_{24}$ | $i \neq 9,10$ | $F_9 = F_{10} \neq 0$ |
| $\|GHZ\rangle_{14} \otimes \|GHZ\rangle_{23}$ | $i \neq 9,10$ | $F_9 = F_{10} \neq 0$ |
| only two qubits are entangled | All $F_i = 0$ | |
| separate states | All $F_i = 0$ | |

Table 6. The properties of $D_i$ for two $GHZ$ pairs

| states | $D_1$ | $D_2$ | $D_3$ |
|---|---|---|---|
| $\|GHZ\rangle_{12} \otimes \|GHZ\rangle_{34}$ | $=0$ | $\neq 0$ | $=0$ |
| $\|GHZ\rangle_{13} \otimes \|GHZ\rangle_{24}$ | $\neq 0$ | $=0$ | $=0$ |
| $\|GHZ\rangle_{14} \otimes \|GHZ\rangle_{23}$ | $=0$ | $=0$ | $\neq 0$ |

## 5 Semi-invariants of $n$-qubits

Definition:

The semi-invariants of a pure state of $n$-qubits is

$$\begin{aligned} F &= 4\Big(\sum_{odd(i+j)} \left|(a_i a_j + a_k a_l - a_p a_q - a_r a_s)^2 - 4(a_i a_{j-1} - a_p a_{q-1})(a_k a_{l+1} - a_r a_{s+1})\right| \\ &\quad + \sum_{Even(i+j)} \left|(a_i a_j + a_k a_l - a_p a_q - a_r a_s)^2 - 4(a_i a_{j-2} - a_p a_{q-2})(a_k a_{l+2} - a_r a_{s+2})\right|\Big), \end{aligned} \tag{12}$$

where

$$\begin{aligned} &i < j, k < l, p < q, r < s, i < k < p < r \\ &i + j = k + l = p + q = r + s, i \oplus j = k \oplus l = p \oplus q = r \oplus s. \end{aligned} \tag{13}$$

For example, $F$ contains the terms in which $i + j = 7, 11, 13, 15, 17, 19$ and 23 and the terms in which $i + j = 14$ and 16, but $F$ does not contain the terms in which $i + j = 8, 9, 10, 12, 18, 20, 21$ or 22.

Remark:

We can consider that Eq. (13) is a special partition of an integer.

Lemma 1.

Let $(|0\rangle + |2^n - 1\rangle)/2$ be state $|GHZ\rangle$ of $n$-qubits. Then $F$ of state $|GHZ\rangle$ does not vanish.

Proof. This is because the following term does not vanish.

$\left|((a_0 a_{2^n-1} - a_2 a_{2^n-3}) + (a_1 a_{2^n-2} - a_3 a_{2^n-4}))^2 - 4(a_0 a_{2^n-2} - a_2 a_{2^n-4})(a_1 a_{2^n-1} - a_3 a_{2^n-3})\right|$
$= |a_0 a_{2^n-1}| = 1/4$.

Notice that other terms vanish.

Lemma 2.

Let $|W\rangle = (|0...01\rangle + |0...010\rangle + |0...0100\rangle + ...)/\sqrt{n}$, where the amplitudes $a_{2^j} = 1/\sqrt{n}$, where $j = 0, 1, ..., (n-1)$, and other amplitudes $a_i = 0$. Then $F$ of $|W\rangle$ vanishes.

Proof.

(1). We show that $a_i a_j = a_k a_l = a_p a_q = a_r a_s = 0$. If $a_i a_j \neq 0$, then $i = 2^m$ and $j = 2^n$, where $m < n$. Clearly, we can not find $k$ or $l$ such that $k = 2^s$ and $l = 2^t$ and $2^m \oplus 2^n = 2^s \oplus 2^t$.

(2). We show that $(a_i a_{j-1} - a_p a_{q-1})(a_k a_{l+1} - a_r a_{s+1}) = 0$. It is enough to illustrate $a_i a_{j-1} = a_p a_{q-1} = a_k a_{l+1} = a_r a_{s+1} = 0$. Assume that $i = 2^m$ and $j - 1 = 2^n$ and $k = 2^s$ and $l + 1 = 2^t$. Since $i + j = k + l$, $2^m + 2^n + 2 = 2^s + 2^t$. Since $i \oplus j = k \oplus l$, then $i \oplus (j-1) = k \oplus (l+1)$, i.e., $2^m \oplus 2^n = 2^s \oplus 2^t$. It is not possible for $2^m + 2^n + 2 = 2^s + 2^t$ and $2^m \oplus 2^n = 2^s \oplus 2$ to both hold.

(3). As well, we can show that $(a_i a_{j-2} - a_p a_{q-2})(a_k a_{l+2} - a_r a_{s+2}) = 0$.

Conclusively, $F$ vanishes.

**Summary**

In this paper, we define the SLOCC invariant and semi-invariants for four-qubits. By means of the invariant and semi-invariants we can determine if two

states are inequivalent. Then, we show that there are infinite $SL$-classes and at least 28 distinct true SLOCC entanglement classes. It seems that there should be more true SLOCC entanglement classes. The invariant and semi-invariants only require simple arithmetic operations. The ideas can be extended to five or more-qubits for SLOCC classification. In this paper, we also give the exact recursive formulas of the number of the degenerated SLOCC classes of $n$-qubits. By the recursive formula, for six-qubits, there are $6*t(5)+30*t(4)+276$ distinct degenerated SLOCC entanglement classes, where $t(5)$ is the number of the true SLOCC entanglement classes for five-qubits.

**Appendix A**

Let $P=\det^2(\beta)\det^2(\delta)\det^2(\gamma)$, $Q=\det^2(\alpha)\det^2(\gamma)\det^2(\delta)$, $R=\det^2(\alpha)\det^2(\beta)\det^2(\delta)$, $S=\det^2(\alpha)\det^2(\gamma)\det^2(\beta)$, $T=\det(\alpha)\det(\beta)\det(\gamma)\det(\delta)$.

We will list $G$, $D_i$ and $F_i$ which are not zero as follows.

1. Class $|GHZ\rangle$

(1). If $F_i=0$ then $|F_9|+|F_{10}|\neq 0$, $i=1,2,3$ and 4.

(2). $IV=-1/2*T$.

Proof of (1):

Let $|\psi'\rangle$ in Eq. (2) be state $|GHZ\rangle$. By solving matrix equation in Eq. (2), we obtain the amplitudes $a_i$. By computing, we obtain the following $F_i$.

$F_1=\frac{1}{4}\alpha_1^2\alpha_2^2P$, $F_2=\frac{1}{4}\alpha_3^2\alpha_4^2P$, $F_3=\frac{1}{4}\beta_1^2\beta_2^2Q$, $F_4=\frac{1}{4}\beta_3^2\beta_4^2Q$, $F_5=\frac{1}{4}\gamma_2^2\gamma_1^2R$, $F_6=\frac{1}{4}\gamma_4^2\gamma_3^2R$, $F_7=\frac{1}{4}\delta_1^2\delta_2^2S$, $F_8=\frac{1}{4}\delta_3^2\delta_4^2S$,

$F_9=\frac{1}{4}\left(\alpha_1\beta_1\alpha_4\beta_4-\beta_3\alpha_3\beta_2\alpha_2\right)^2\det^2(\delta)\det^2(\gamma)$,

$F_{10}=\frac{1}{4}\left(-\alpha_1\beta_3\alpha_4\beta_2+\beta_1\alpha_3\beta_4\alpha_2\right)^2\det^2(\delta)\det^2(\gamma)$.

Assume $F_1=0$ and $F_3=0$. Then $\alpha_1\alpha_2=0$. Without loss of generality, let us consider $\alpha_1=0$. This implies $\alpha_2\alpha_3\neq 0$ since $\alpha$ is invertible. Thus, $F_9=\frac{1}{4}\left(\beta_2\beta_3\alpha_2\alpha_3\right)^2\det^2(\delta)\det^2(\gamma)$ and $F_{10}=\frac{1}{4}\left(\beta_1\beta_4\alpha_2\alpha_3\right)^2\det^2(\delta)\det^2(\gamma)$. If $\beta_1=0$ then $F_{10}=0$ and $F_9\neq 0$ because $\beta$ is invertible. If $\beta_2=0$, then $F_9=0$ and $F_{10}\neq 0$. Similarly, it is easy to verify other cases.

2. Class $|C_4\rangle$

(1). If $F_i=F_j=F_k=0$, where $1\leq i<j<k\leq 4$, then $|F_9|+|F_{10}|\neq 0$ and $F_9F_{10}=0$.

(2). $IV=-1/2*T$.

$D_1=(-1/36)((\alpha_2\alpha_3+\alpha_1\alpha_4)(\gamma_2\gamma_3+\gamma_1\gamma_4)+\alpha_2\alpha_4\gamma_1\gamma_3+\alpha_1\alpha_3\gamma_2\gamma_4)\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_2=(1/36)((\alpha_2\alpha_3+\alpha_1\alpha_4)(\beta_2\beta_3+\beta_1\beta_4)+\alpha_2\alpha_4\beta_1\beta_3+\alpha_1\alpha_3\beta_2\beta_4)\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$D_3=(1/36)((\alpha_2\alpha_3+\alpha_1\alpha_4)(\delta_2\delta_3+\delta_1\delta_4)+\alpha_2\alpha_4\delta_1\delta_3+\alpha_1\alpha_3\delta_2\delta_4)\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

Proof of (1):

Let $|\psi'\rangle$ in Eq. (2) be state $|C_4\rangle$. By solving matrix equation in Eq. (2), we obtain the amplitudes $a_i$. Then, we obtain the following $F_i$.

$F_1$ to $F_8$ can be obtained from the ones of class $|GHZ\rangle$ above by replacing 1/4 by $(-1/12)$, respectively.

$F_9=(1/36)(-4\alpha_2^2\alpha_3\alpha_4\beta_1\beta_2\beta_3^2+4\alpha_1\alpha_2\alpha_4^2\beta_1\beta_2\beta_3^2+\alpha_2^2\alpha_3^2\beta_2^2\beta_3^2-4\alpha_1\alpha_2\alpha_3\alpha_4\beta_2^2\beta_3^2+4\alpha_2^2\alpha_3\alpha_4\beta_1^2\beta_3\beta_4-4\alpha_1\alpha_2\alpha_4^2\beta_1^2\beta_3\beta_4-4\alpha_2^2\alpha_3^2\beta_1\beta_2\beta_3\beta_4+14\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4-4\alpha_1^2\alpha_4^2\beta_1\beta_2\beta_3\beta_4-4\alpha_1\alpha_2\alpha_3^2\beta_2^2\beta_3\beta_4+4\alpha_1^2\alpha_3\alpha_4\beta_2^2\beta_3\beta_4-4\alpha_1\alpha_2\alpha_3\alpha_4\beta_1^2\beta_4^2+\alpha_1^2\alpha_4^2\beta_1^2\beta_4^2+4\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2\beta_4^2-4\alpha_1^2\alpha_3\alpha_4\beta_1\beta_2\beta_4^2)\det^2(\delta)\det^2(\gamma)$,

$F_{10} = (1/36)(4\alpha_2^2\alpha_3\alpha_4\beta_1\beta_2\beta_3^2 - 4\alpha_1\alpha_2\alpha_4^2\beta_1\beta_2\beta_3^2 - 4\alpha_1\alpha_2\alpha_3\alpha_4\beta_2^2\beta_3^2 + \alpha_1^2\alpha_4^2\beta_2^2\beta_3^2 - 4\alpha_2^2\alpha_3\alpha_4\beta_1^2\beta_3\beta_4 + 4\alpha_1\alpha_2\alpha_4^2\beta_1^2\beta_3\beta_4 - 4\alpha_2^2\alpha_3^2\beta_1\beta_2\beta_3\beta_4 + 14\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4 - 4\alpha_1^2\alpha_4^2\beta_1\beta_2\beta_3\beta_4 + 4\alpha_1\alpha_2\alpha_3^2\beta_2^2\beta_3\beta_4 - 4\alpha_1^2\alpha_3\alpha_4\beta_2^2\beta_3\beta_4 + \alpha_2^2\alpha_3^2\beta_1^2\beta_4^2 - 4\alpha_1\alpha_2\alpha_3\alpha_4\beta_1^2\beta_4^2 - 4\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2\beta_4^2 + 4\alpha_1^2\alpha_3\alpha_4\beta_1\beta_2\beta_4^2)\det^2(\delta)\det^2(\gamma)$.

Let us prove that if $F_1 = F_3 = F_4 = 0$ then $|F_9| + |F_{10}| \neq 0$ and $F_9F_{10} = 0$. The proofs for other cases are similar.

Assume that $F_1 = 0$. Then there are two cases: case 1, $\alpha_1 = 0$ and case 2, $\alpha_2 = 0$.

Case 1. $\alpha_1 = 0$. In this case, $\alpha_2\alpha_3 \neq 0$. Thus,

$F_9 = (1/36)\alpha_2^2\alpha_3\beta_3\left(\alpha_3\beta_2^2\beta_3 + 4\beta_1^2\alpha_4\beta_4 - 4\beta_1\alpha_3\beta_2\beta_4 - 4\beta_1\beta_2\alpha_4\beta_3\right)\det^2(\delta)\det^2(\gamma)$,

$F_{10} = (1/36)\alpha_2^2\beta_1\alpha_3\left(\beta_1\alpha_3\beta_4^2 + 4\beta_2\alpha_4\beta_3^2 - 4\beta_1\alpha_4\beta_3\beta_4 - 4\alpha_3\beta_2\beta_3\beta_4\right)\det^2(\delta)\det^2(\gamma)$.

Since $F_3 = F_4 = 0$, there are two cases.

Case 1.1. $\beta_1 = \beta_4 = 0$. Then $\beta_2\beta_3 \neq 0$.

$F_9 = (1/36)\alpha_2^2\alpha_3^2\beta_2^2\beta_3^2\det^2(\delta)\det^2(\gamma) \neq 0$,

$F_{10} = 0$.

Case 1.2. $\beta_2 = \beta_3 = 0$. Then $\beta_1\beta_4 \neq 0$.

$F_9 = 0$,

$F_{10} = (1/36)\alpha_2^2\alpha_3^2\left(\beta_1^2\beta_4^2\right)\det^2(\delta)\det^2(\gamma) \neq 0$.

Case 2. $\alpha_2 = 0$. In this case, $\alpha_1\alpha_4 \neq 0$. Thus,

$F_9 = (1/36)\alpha_1^2\alpha_4\beta_4\left(4\alpha_3\beta_2^2\beta_3 + \beta_1^2\alpha_4\beta_4 - 4\beta_1\alpha_3\beta_2\beta_4 - 4\beta_1\beta_2\alpha_4\beta_3\right)\det^2(\delta)\det^2(\gamma)$,

$F_{10} = (1/36)\alpha_1^2\beta_2\alpha_4\left(4\beta_1\alpha_3\beta_4^2 + \beta_2\alpha_4\beta_3^2 - 4\beta_1\alpha_4\beta_3\beta_4 - 4\alpha_3\beta_2\beta_3\beta_4\right)\det^2(\delta)\det^2(\gamma)$.

Since $F_3 = F_4 = 0$, there are two cases.

Case 1.1. $\beta_1 = \beta_4 = 0$. Then $\beta_2\beta_3 \neq 0$.

$F_9 = 0$,

$F_{10} = (1/36)\alpha_1^2\alpha_4^2\beta_2^2\beta_3^2\det^2(\delta)\det^2(\gamma) \neq 0$.

Case 1.2. $\beta_2 = \beta_3 = 0$. Then $\beta_1\beta_4 \neq 0$.

$F_9 = (1/36)\alpha_1^2\alpha_4^2\left(\beta_1^2\beta_4^2\right)\det^2(\delta)\det^2(\gamma) \neq 0$,

$F_{10} = 0$.

3. Class $|\kappa_4\rangle$

(1). $IV = 1/4 * T$.

$D_1 = (1/16)\alpha_2\alpha_4\gamma_2\gamma_4\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_2 = (1/16)\alpha_1\alpha_3\beta_1\beta_3\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

(2).#0

Proof of (2):

$F_i$ can be obtained from the $F_i$ of class $|GHZ\rangle$ by replacing 1/4 by 1/16. So the proof is similar to the one of class $|GHZ\rangle$.

4. Class $|E_4\rangle$

(1). $IV = 1/4 * T$.

$D_1 = -(1/16)\alpha_1\alpha_3\gamma_1\gamma_3\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_3 = -(1/16)\alpha_2\alpha_4\delta_2\delta_4\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). #0

Proof of (2):

$F_i$ are the same as the $F_i$ of class $|\kappa_4\rangle$. So the proof is similar to the one of $|\kappa_4\rangle$.

5. Class $|L_4\rangle$

(1). $IV = 1/4 * T$.

$D_2 = (1/16)\alpha_1\alpha_3\beta_1\beta_3 \det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$D_3 = -(1/16)\alpha_2\alpha_4\delta_2\delta_4 \det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). #0

Proof of (2):

$F_i$ are the same as the $F_i$ of class $|\kappa_4\rangle$.

6. Class $|H_4\rangle$

(1). $IV = -1/3 * T$. $D_1 = -(1/9)\alpha_1\alpha_3\gamma_2\gamma_4 \det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$.

(2). #0

Proof of (2):

$F_i$ can be obtained from the $F_i$ of class $|GHZ\rangle$ by replacing 1/4 by 1/9.

7. Class $|\lambda_4\rangle$

(1). $IV = -1/3 * T$. $D_2 = (1/9)\alpha_1\alpha_3\beta_2\beta_4 \det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

(2). #0

Proof of (2):

$F_1$ to $F_8$ can be obtained from $F_1$ to $F_8$ of class $|GHZ\rangle$ by replacing 1/4 by 1/9, respectively.

$F_9 = (1/9)\left(\alpha_1\beta_2\alpha_4\beta_3 - \beta_1\alpha_2\beta_4\alpha_3\right)^2 \det^2(\gamma)\det^2(\delta)$,

$F_{10} = (1/9)\left(-\alpha_1\beta_1\alpha_4\beta_4 + \beta_2\alpha_3\beta_3\alpha_2\right)^2 \det^2(\gamma)\det^2(\delta)$.

The next argument is the same as the one for class $|GHZ\rangle$.

8. Class $M_4$

(1). $IV = -1/3 * T$. $D_3 = (1/9)\alpha_1\alpha_3\delta_2\delta_4 \det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). #0

Proof of (2):

$F_i$ are the same as the ones of class $|H_4\rangle$.

9. Class $|\pi_4\rangle$

(1). $|F_1| + |F_2| \neq 0, |F_5| + |F_6| \neq 0$, #2.

(2). $IV = -1/3 * T$.

$D_1 = \frac{1}{36}(2\alpha_1\alpha_3\gamma_1\gamma_3 + \alpha_2\alpha_3\gamma_2\gamma_3 + \alpha_1\alpha_4\gamma_2\gamma_3 + \alpha_2\alpha_3\gamma_1\gamma_4 + \alpha_1\alpha_4\gamma_1\gamma_4 + 2\alpha_1\alpha_3\gamma_2\gamma_4 + 2\alpha_2\alpha_4\gamma_2\gamma_4)\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$

Proof of (1):

Let $|\psi'\rangle$ in Eq. (2) be state $|\pi_4\rangle$. By solving matrix equation in Eq. (2), we obtain the amplitudes $a_i$. Then, we obtain the following $F_i$.

$F_1 = \alpha_1^4 P/9$, $F_2 = \alpha_3^4 P/9$, $F_5 = \gamma_2^4 R/9$, $F_6 = \gamma_4^4 R/9$,

$F_9 = \det(\beta)(-4\alpha_1\alpha_2\alpha_3^2\beta_1\beta_3 + 4\alpha_1^2\alpha_3\alpha_4\beta_1\beta_3 - 4\alpha_1^2\alpha_3^2\beta_2\beta_3 - \alpha_2^2\alpha_3^2\beta_2\beta_3 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_2\beta_3 - \alpha_1^2\alpha_4^2\beta_2\beta_3 + 4\alpha_1^2\alpha_3^2\beta_1\beta_4 + \alpha_2^2\alpha_3^2\beta_1\beta_4 - 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_4 + \alpha_1^2\alpha_4^2\beta_1\beta_4 + 4\alpha_1\alpha_2\alpha_3^2\beta_2\beta_4 - 4\alpha_1^2\alpha_3\alpha_4\beta_2\beta_4)\det^2(\gamma)\det^2(\delta)/36$,

$F_{10} = \det(\beta)(4\alpha_1\alpha_2\alpha_3^2\beta_1\beta_3 - 4\alpha_1^2\alpha_3\alpha_4\beta_1\beta_3 - 4\alpha_1^2\alpha_3^2\beta_2\beta_3 - \alpha_2^2\alpha_3^2\beta_2\beta_3 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_2\beta_3 - \alpha_1^2\alpha_4^2\beta_2\beta_3 + 4\alpha_1^2\alpha_3^2\beta_1\beta_4 + \alpha_2^2\alpha_3^2\beta_1\beta_4 - 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_4 + \alpha_1^2\alpha_4^2\beta_1\beta_4 - 4\alpha_1\alpha_2\alpha_3^2\beta_2\beta_4 + 4\alpha_1^2\alpha_3\alpha_4\beta_2\beta_4)\det^2(\gamma)\det^2(\delta)/36$.

Clearly, if $F_1 = 0$, then $\alpha_1 = 0$. Since $\alpha$ is invertible, that $\alpha_1 = 0$ implies $\alpha_3 \neq 0$. Thus, $F_2 \neq 0$. Therefore $|F_1| + |F_2| \neq 0$. As well, $|F_5| + |F_6| \neq 0$.

Next we prove if $F_1F_2 = 0$ then $F_9 = F_{10} \neq 0$. Assume $F_1 = 0$. Then $\alpha_1 = 0$. By substituting $\alpha_1 = 0$ into $F_9$ and $F_{10}$, $F_9 = F_{10} = \alpha_2^2\alpha_3^2(\beta_1\beta_4 - \beta_2\beta_3)/36$.

Since $\alpha$ and $\beta$ are invertible, $F_9 = F_{10} \neq 0$. Similarly, we can argue $F_9 = F_{10} \neq 0$ if $F_2 = 0$.

10. Class $|\theta_4\rangle$

(1). $IV = -1/3 * T$.

$D_1 = \frac{1}{36}(2\alpha_1\alpha_3\gamma_1\gamma_3 + \alpha_2\alpha_3\gamma_2\gamma_3 + \alpha_1\alpha_4\gamma_2\gamma_3 + \alpha_2\alpha_3\gamma_1\gamma_4 + \alpha_1\alpha_4\gamma_1\gamma_4 + 2\alpha_2\alpha_4\gamma_2\gamma_4)\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$.

(2). $|F_3| + |F_4| \neq 0, |F_7| + |F_8| \neq 0, \#3$.

Proof.

Let $|\psi'\rangle$ in Eq. (2) be state $|\theta_4\rangle$. By solving matrix equation in Eq. (2), we obtain the amplitudes $a_i$. Then, we obtain the following $F_i$.

$F_3 = \frac{1}{9}\beta_2^4 Q$, $F_4 = \frac{1}{9}\beta_4^4 Q$, $F_7 = \frac{1}{9}\delta_1^4 S$, $F_8 = \frac{1}{9}\delta_3^4 S$,

$F_9 = \frac{1}{36}(-\alpha_2\alpha_3\beta_2^2\beta_3^2 + \alpha_1\alpha_4\beta_2^2\beta_3^2 + 2\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4 - 2\alpha_1\alpha_4\beta_1\beta_2\beta_3\beta_4 + 4\alpha_1\alpha_3\beta_2^2\beta_3\beta_4 - 4\alpha_2\alpha_4\beta_2^2\beta_3\beta_4 - \alpha_2\alpha_3\beta_1^2\beta_4^2 + \alpha_1\alpha_4\beta_1^2\beta_4^2 - 4\alpha_1\alpha_3\beta_1\beta_2\beta_4^2 + 4\alpha_2\alpha_4\beta_1\beta_2\beta_4^2 - 4\alpha_2\alpha_3\beta_2^2\beta_4^2 + 4\alpha_1\alpha_4\beta_2^2\beta_4^2)\det(\alpha)\det^2(\gamma)\det^2(\delta)$,

$F_{10} = \frac{1}{36}(-\alpha_2\alpha_3\beta_2^2\beta_3^2 + \alpha_1\alpha_4\beta_2^2\beta_3^2 + 2\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4 - 2\alpha_1\alpha_4\beta_1\beta_2\beta_3\beta_4 - 4\alpha_1\alpha_3\beta_2^2\beta_3\beta_4 + 4\alpha_2\alpha_4\beta_2^2\beta_3\beta_4 - \alpha_2\alpha_3\beta_1^2\beta_4^2 + \alpha_1\alpha_4\beta_1^2\beta_4^2 + 4\alpha_1\alpha_3\beta_1\beta_2\beta_4^2 - 4\alpha_2\alpha_4\beta_1\beta_2\beta_4^2 - 4\alpha_2\alpha_3\beta_2^2\beta_4^2 + 4\alpha_1\alpha_4\beta_2^2\beta_4^2)\det(\alpha)\det^2(\gamma)\det^2(\delta)$.

The proofs of the properties for $F_i$ are similar to the ones of class $|\pi_4\rangle$. Next we prove if $F_3F_4 = 0$ then $F_9 = F_{10} \neq 0$. Assume $F_3 = 0$. Then $\beta_2 = 0$. Then $F_9 = F_{10} = 1/36(\beta_1^2\beta_4^2(\alpha_1\alpha_4 - \alpha_2\alpha_3) \neq 0$ because $\alpha$ and $\beta$ are invertible. As well, we can show if $F_4 = 0$ then $F_9 = F_{10} \neq 0$.

11. Class $|\sigma_4\rangle$

(1). $IV = 1/3 * T$.

$D_2 = -\frac{1}{36}(2\alpha_1\alpha_3\beta_1\beta_3 + 2\alpha_2\alpha_4\beta_1\beta_3 + \alpha_2\alpha_3\beta_2\beta_3 + \alpha_1\alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_1\beta_4 + \alpha_1\alpha_4\beta_1\beta_4 + 2\alpha_2\alpha_4\beta_2\beta_4)\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta\det(\alpha)\det(\alpha))$.

(2). $|F_1| + |F_2| \neq 0, |F_3| + |F_4| \neq 0$, if $F_1F_2 = 0$ and $F_3F_4 = 0$ then $F_9 = 0 \wedge F_{10} = 0$.

Proof.

$F_1 = (1/9)\alpha_2^4 P$, $F_2 = (1/9)\alpha_4^4 P$, $F_3 = (1/9)\beta_1^4 Q$, $F_4 = (1/9)\beta_3^4 Q$,

$F_9 = (1/9)(\alpha_2\alpha_3\beta_1\beta_3 - \alpha_1\alpha_4\beta_1\beta_3 - \alpha_2\alpha_4\beta_2\beta_3 + \alpha_2\alpha_4\beta_1\beta_4)^2\det^2(\delta)\det^2(\gamma)$,

$F_{10} = (1/9)(\alpha_2\alpha_3\beta_1\beta_3 - \alpha_1\alpha_4\beta_1\beta_3 + \alpha_2\alpha_4\beta_2\beta_3 - \alpha_2\alpha_4\beta_1\beta_4)^2\det^2(\delta)\det^2(\gamma)$.

12. Class $|\rho_4\rangle$

(1). $IV = 1/3 * T$. $D_2 = -\frac{1}{36}(2\alpha_1\alpha_3\beta_1\beta_3 + \alpha_2\alpha_3\beta_2\beta_3 + \alpha_1\alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_1\beta_4 + \alpha_1\alpha_4\beta_1\beta_4 + 2\alpha_2\alpha_4\beta_2\beta_4)\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

(2). $F_5 = \frac{1}{9}\gamma_2^4 R$, $F_6 = \frac{1}{9}\gamma_4^4 R$, $F_7 = \frac{1}{9}\delta_1^4 S$, $F_8 = \frac{1}{9}\delta_3^4 S$.

13. Class $|\xi_4\rangle$

(1). $IV = -1/3 * T$.

$D_3 = -\frac{1}{36}(2\alpha_1\alpha_3\delta_1\delta_3 + 2\alpha_2\alpha_4\delta_1\delta_3 + \alpha_2\alpha_3\delta_2\delta_3 + \alpha_1\alpha_4\delta_2\delta_3 + \alpha_2\alpha_3\delta_1\delta_4 + \alpha_1\alpha_4\delta_1\delta_4 + 2\alpha_2\alpha_4\delta_2\delta_4)\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$,

(2). $|F_1| + |F_2| \neq 0, |F_7| + |F_8| \neq 0, \#2$.

Proof of (2):

$F_1 = \frac{1}{9}\alpha_2^4 P$, $F_2 = \frac{1}{9}\alpha_4^4 P$, $F_7 = \frac{1}{9}\delta_1^4 S$, $F_8 = \frac{1}{9}\delta_3^4 S$,

$F_9 = \frac{1}{36}(4\alpha_2^2\alpha_3\alpha_4\beta_1\beta_3 - 4\alpha_1\alpha_2\alpha_4^2\beta_1\beta_3 - \alpha_2^2\alpha_3^2\beta_2\beta_3 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_2\beta_3 - \alpha_1^2\alpha_4^2\beta_2\beta_3 - 4\alpha_2^2\alpha_4^2\beta_2\beta_3 + \alpha_2^2\alpha_3^2\beta_1\beta_4 - 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_4 + \alpha_1^2\alpha_4^2\beta_1\beta_4 + 4\alpha_2^2\alpha_4^2\beta_1\beta_4 - 4\alpha_2^2\alpha_3\alpha_4\beta_2\beta_4 + 4\alpha_1\alpha_2\alpha_4^2\beta_2\beta_4)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$F_{10} = \frac{1}{36}(-4\alpha_2^2\alpha_3\alpha_4\beta_1\beta_3 + 4\alpha_1\alpha_2\alpha_4^2\beta_1\beta_3 - \alpha_2^2\alpha_3^2\beta_2\beta_3 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_2\beta_3 - \alpha_1^2\alpha_4^2\beta_2\beta_3 - 4\alpha_2^2\alpha_4^2\beta_2\beta_3 + \alpha_2^2\alpha_3^2\beta_1\beta_4 - 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_4 + \alpha_1^2\alpha_4^2\beta_1\beta_4 + 4\alpha_2^2\alpha_4^2\beta_1\beta_4 + 4\alpha_2^2\alpha_3\alpha_4\beta_2\beta_4 - 4\alpha_1\alpha_2\alpha_4^2\beta_2\beta_4)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

The rest proofs are similar to the ones of $|\pi_4\rangle$.

14. Class $|\epsilon_4\rangle$

(1). $IV = -1/3 * T$.

$D_3 = -\frac{1}{36}(2\alpha_1\alpha_3\delta_1\delta_3 + \alpha_2\alpha_3\delta_2\delta_3 + \alpha_1\alpha_4\delta_2\delta_3 + \alpha_2\alpha_3\delta_1\delta_4 + \alpha_1\alpha_4\delta_1\delta_4 + 2\alpha_2\alpha_4\delta_2\delta_4)\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). $|F_3| + |F_4| \neq 0, |F_5| + |F_6| \neq 0$, #3.

Proof.

$F_3 = \frac{1}{9}\beta_1^4 Q$, $F_4 = \frac{1}{9}\beta_3^4 Q$, $F_5 = \frac{1}{9}\gamma_2^4 R$, $F_6 = \frac{1}{9}\gamma_4^4 R$,

$F_9 = \frac{1}{36}(-4\alpha_2\alpha_3\beta_1^2\beta_3^2 + 4\alpha_1\alpha_4\beta_1^2\beta_3^2 - 4\alpha_1\alpha_3\beta_1\beta_2\beta_3^2 + 4\alpha_2\alpha_4\beta_1\beta_2\beta_3^2 - \alpha_2\alpha_3\beta_2^2\beta_3^2 + \alpha_1\alpha_4\beta_2^2\beta_3^2 + 4\alpha_1\alpha_3\beta_1^2\beta_3\beta_4 - 4\alpha_2\alpha_4\beta_1^2\beta_3\beta_4 + 2\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4 - 2\alpha_1\alpha_4\beta_1\beta_2\beta_3\beta_4 - \alpha_2\alpha_3\beta_1^2\beta_4^2 + \alpha_1\alpha_4\beta_1^2\beta_4^2)\det(\alpha)\det^2(\gamma)\det^2(\delta)$,

$F_{10} = -\frac{1}{36}(4\alpha_2\alpha_3\beta_1^2\beta_3^2 - 4\alpha_1\alpha_4\beta_1^2\beta_3^2 - 4\alpha_1\alpha_3\beta_1\beta_2\beta_3^2 + 4\alpha_2\alpha_4\beta_1\beta_2\beta_3^2 + \alpha_2\alpha_3\beta_2^2\beta_3^2 - \alpha_1\alpha_4\beta_2^2\beta_3^2 + 4\alpha_1\alpha_3\beta_1^2\beta_3\beta_4 - 4\alpha_2\alpha_4\beta_1^2\beta_3\beta_4 - 2\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4 + 2\alpha_1\alpha_4\beta_1\beta_2\beta_3\beta_4 + \alpha_2\alpha_3\beta_1^2\beta_4^2 - \alpha_1\alpha_4\beta_1^2\beta_4^2)\det(\alpha)\det^2(\gamma)\det^2(\delta)$.

The rest proofs of the properties for $F_i$ are similar to the ones of $|\theta_4\rangle$.

15. Class $|\chi_4\rangle$

(1). $D_1 = -(1/18)(\alpha_1\alpha_3 - \alpha_2\alpha_4)\gamma_2\gamma_4\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_2 = (1/36)(\alpha_2\alpha_3 + \alpha_1\alpha_4)(\beta_2\beta_3 + \beta_1\beta_4)\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$D_3 = (1/18)(\alpha_1\alpha_3 + \alpha_2\alpha_4)\delta_1\delta_3\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). $|F_5| + |F_6| \neq 0$, $|F_7| + |F_8| \neq 0$, #0.

Proof of (2):

$F_1 = \frac{1}{9}\alpha_1^2\alpha_2^2 P$, $F_2 = \frac{1}{9}\alpha_3^2\alpha_4^2 P$, $F_3 = \frac{1}{9}\beta_1^2\beta_2^2 Q$, $F_4 = \frac{1}{9}\beta_3^2\beta_4^2 Q$, $F_5 = \frac{1}{9}(\gamma_1 - \gamma_2)\gamma_2^2(\gamma_1 + \gamma_2)R$, $F_6 = \frac{1}{9}(\gamma_3 - \gamma_4)\gamma_4^2(\gamma_3 + \gamma_4)R$, $F_7 = \frac{1}{9}\delta_1^2(\delta_1^2 + \delta_2^2)S$, $F_8 = \frac{1}{9}\delta_3^2(\delta_3^2 + \delta_4^2)S$,

$F_9 = \frac{1}{9}\left(\alpha_1\beta_1\alpha_4\beta_4 - \beta_3\alpha_3\beta_2\alpha_2\right)^2\det^2(\delta)\det^2(\gamma)$,

$F_{10} = \frac{1}{9}\left(-\alpha_1\beta_3\alpha_4\beta_2 + \beta_1\alpha_3\beta_4\alpha_2\right)^2\det^2(\delta)\det^2(\gamma)$.

Let us show $|F_5| + |F_6| \neq 0$. Case 1, $\gamma_2 = 0$. Then $\gamma_4 \neq 0$ because $\gamma$ is invertible. Next we show $\gamma_3^2 \neq \gamma_4^2$. Otherwise, $\gamma_4 = 0$ when $\gamma_3 = 0$. This contradicts that $\gamma$ is invertible. Case 2, $\gamma_4 = 0$. Similarly, we can show $\gamma_2 \neq 0$ and $\gamma_1^2 \neq \gamma_2^2$. Case 3, $\gamma_1^2 = \gamma_2^2$. Then $\gamma_3^2 \neq \gamma_4^2$. Otherwise, $\gamma_1^2\gamma_4^2 = \gamma_2^2\gamma_3^2$. Since $\gamma_1\gamma_4 - \gamma_2\gamma_3 \neq 0$, $\gamma_1\gamma_4 + \gamma_2\gamma_3 = 0$. Thus, $\det(\gamma) = -2\gamma_2\gamma_3$. Clearly, for an invertible $\gamma$ in which $\gamma_2\gamma_3 = 0$, $\det(\gamma) = 0$. This a paradox. Case 4, $\gamma_3^2 = \gamma_4^2$. Similarly, $\gamma_1^2 \neq \gamma_2^2$.

The rest proof is similar to the one of $|GHZ\rangle$.

16. Class $|\upsilon_4\rangle$

(1). $D_2 = \frac{1}{16}\alpha_1\alpha_3\beta_1\beta_3\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$D_3 = -\frac{1}{16}\alpha_1\alpha_3\delta_2\delta_4\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). #1.

$F_1 = \frac{1}{16}\alpha_1^4 P$, $F_2 = \frac{1}{16}\alpha_3^4 P$, $F_3 = \frac{1}{16}\beta_1^4 Q$, $F_4 = \frac{1}{16}\beta_3^4 Q$, $F_7 = \frac{1}{16}\delta_2^4 S$, $F_8 = \frac{1}{16}\delta_4^4 S$,

$F_9 = \frac{1}{16}(\alpha_2\alpha_3\beta_1\beta_3 - \alpha_1\alpha_4\beta_1\beta_3 + \alpha_1\alpha_3\beta_2\beta_3 - \alpha_1\alpha_3\beta_1\beta_4)^2 \det^2(\gamma)\det^2(\delta)$,

$F_{10} = \frac{1}{16}(-\alpha_2\alpha_3\beta_1\beta_3 + \alpha_1\alpha_4\beta_1\beta_3 + \alpha_1\alpha_3\beta_2\beta_3 - \alpha_1\alpha_3\beta_1\beta_4)^2 \det^2(\gamma)\det^2(\delta)$.

17. Class $|\varpi_4\rangle$

(1). $D_1 = -\frac{1}{16}\alpha_1\alpha_3\gamma_1\gamma_3 \det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_3 = \frac{1}{16}\alpha_1\alpha_3\delta_1\delta_3 \det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). $F_1 = \frac{1}{16}\alpha_1^4 P$, $F_2 = \frac{1}{16}\alpha_3^4 P$, $F_5 = \frac{1}{16}\gamma_1^4 R$, $F_6 = \frac{1}{16}\gamma_3^4 R$, $F_7 = \frac{1}{16}\delta_1^4 S$, $F_8 = \frac{1}{16}\delta_3^4 S$, $F_9 = \frac{1}{16}\alpha_1^2\alpha_3^2 P$, $F_{10} = F_9$.

18. Class $|\psi_4\rangle$

(1). $F > 0$, $F_9 = F_{10}$, if $F_i = F_j = F_k = 0$, where $1 \le i < j < k \le 4$, then $F_9 \ne 0$.

(2). $D_1 = (-1/16)(\alpha_2\alpha_3 + \alpha_1\alpha_4)(\gamma_2\gamma_3 + \gamma_1\gamma_4)\det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$.

Proof of (1):

The values of $F_1$ to $F_8$ are the same as the ones of $F_1$ to $F_8$ of class $|GHZ\rangle$, respectively.

$F_9 = (1/16)(\alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_2^2\beta_3^2 + \alpha_1^2\alpha_4^2\beta_2^2\beta_3^2 + 2\alpha_2^2\alpha_3^2\beta_1\beta_2\beta_3\beta_4 - 12\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4 + 2\alpha_1^2\alpha_4^2\beta_1\beta_2\beta_3\beta_4 + \alpha_2^2\alpha_3^2\beta_1^2\beta_4^2 + 2\alpha_1\alpha_2\alpha_3\alpha_4\beta_1^2\beta_4^2 + \alpha_1^2\alpha_4^2\beta_1^2\beta_4^2)\det^2(\gamma)\det^2(\beta)$,

$F_{10} = F_9$.

Let us prove that if $F_1 = F_3 = F_4 = 0$ then $F_9 \ne 0$. The proofs for other cases are similar.

Assume that $F_1 = 0$. Then there are two cases: case 1, $\alpha_1 = 0$ and case 2, $\alpha_2 = 0$.

Case 1. $\alpha_1 = 0$. In this case, $\alpha_2\alpha_3 \ne 0$. Thus,

$F_9 = (1/16)\alpha_2^2\alpha_3^2(\beta_1\beta_4 + \beta_2\beta_3)^2 \det^2(\gamma)\det^2(\beta)$.

Since $F_3 = F_4 = 0$, there are two cases.

Case 1.1 $\beta_1 = \beta_4 = 0$. Then $\beta_2\beta_3 \ne 0$.

Case 1.2. $\beta_2 = \beta_3 = 0$. Then $\beta_1\beta_4 \ne 0$.

In either case, it is straightforward that $F_9 \ne 0$.

Case 2. $\alpha_2 = 0$. In this case, $\alpha_1\alpha_4 \ne 0$. Thus,

$F_9 = (1/16)\alpha_1^2\alpha_4^2(\beta_1\beta_4 + \beta_2\beta_3)^2 \det^2(\gamma)\det^2(\beta)$.

As discussed above, when $F_3 = F_4 = 0$, $F_9 \ne 0$.

19. Class $|\phi_4\rangle$

(1). $F > 0$.

(2). $D_2 = (1/16)(\alpha_2\alpha_3 + \alpha_1\alpha_4)(\beta_2\beta_3 + \beta_1\beta_4)\det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

(3). #0.

Proof.

$F_1$ to $F_{10}$ are the same as $F_1$ to $F_{10}$ of class $|GHZ\rangle$. The next argument is the same as the one of class $|GHZ\rangle$.

20. Class $|\mu_4\rangle$

(1). $F_9 = F_{10}$, if $F_i = F_j = F_k = 0$, where $1 \le i < j < k \le 4$, then $F_9 \ne 0$.

(2). $D_3 = (1/16)(\alpha_2\alpha_3 + \alpha_1\alpha_4)(\delta_2\delta_3 + \delta_1\delta_4)\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

$F_1$ to $F_{10}$ are the same as the ones of class $|\psi_4\rangle$. The argument is the same as the one for class $|\psi_4\rangle$.

21. Class $|\varphi_4\rangle$

(1) $D_1 = \frac{1}{9}\alpha_1\alpha_3\gamma_2\gamma_4 \det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$.

(2). $F_1 = \frac{1}{9}\alpha_1^4 P$, $F_2 = \frac{1}{9}\alpha_3^4 P$, $F_5 = \frac{1}{9}\gamma_2^4 R$, $F_6 = \frac{1}{9}\gamma_4^4 R$, $F_9 = \frac{1}{9}\alpha_1^2\alpha_3^2 P$, $F_{10} = F_9$.

22. Class $|\zeta_4\rangle$

(1) $D_2 = -\frac{1}{9}\alpha_2\alpha_4\beta_1\beta_3 \det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$.

(2). #1.

The values of $F_i$ are the same as the ones of $F_i$ for class $|\sigma_4\rangle$.

23. Class $|\vartheta_4\rangle$

(1) $D_3 = -\frac{1}{9}\alpha_1\alpha_3\delta_2\delta_4 \det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)$.

(2). $F_1 = \frac{1}{9}\alpha_1^4 P$, $F_2 = \frac{1}{9}\alpha_3^4 P$, $F_7 = \frac{1}{9}\delta_2^4 S$, $F_8 = \frac{1}{9}\delta_4^4 S$, $F_9 = \frac{1}{9}\alpha_1^2\alpha_3^2 P$, $F_{10} = F_9$.

24. Class $|\tau_4\rangle$

$F_3 = \frac{1}{9}\beta_1^4 Q$, $F_4 = \frac{1}{9}\beta_3^4 Q$, $F_5 = \frac{1}{9}\gamma_2^4 R$, $F_6 = \frac{1}{9}\gamma_4^4 R$, $F_9 = \frac{1}{9}\beta_1^2\beta_3^2 Q$, $F_{10} = F_9$.

25. Class $|\varrho_4\rangle$

$F_3 = \frac{1}{9}\beta_1^4 Q$, $F_4 = \frac{1}{9}\beta_3^4 Q$, $F_7 = \frac{1}{9}\delta_2^4 S$, $F_8 = \frac{1}{9}\delta_4^4 S$, $F_9 = \frac{1}{9}\beta_1^2\beta_3^2 Q$, $F_{10} = F_9$.

26. Class $|\iota_4\rangle$

$F_5 = \gamma_1^4 R/9$, $F_6 = \gamma_3^4 R/9$, $F_7 = \delta_1^4 S/9$, $F_8 = \delta_3^4 S/9$.

27. Class $|\omega_4\rangle$

$F_3 = \beta_2^4 Q/16$, $F_4 = \beta_4^4 Q/16$, $F_5 = \gamma_1^4 R/16$, $F_6 = \gamma_3^4 R/16$, $F_7 = \delta_1^4 S/16$, $F_8 = \delta_3^4 S/16$, $F_9 = F_{10} = \beta_2^2\beta_4^2 Q/16$.

**Appendix B: The number of the degenerated SLOCC classes for $n$-qubits**

Let $d(n)$ be the number of the degenerated SLOCC classes for $n$-qubits and $t(n)$ the number of the true entanglement SLOCC classes for $n$-qubits and $t(1) = 1$.

1. **By computing $d(5)$ demonstrate how to derive $d(n)$**

Case 1. Only 4-qubit true entanglement accompanied with a separable qubit

For example, $ABCD - E$, where $ABCD$ is truly entangled. Note that four-qubits can truly be entangled in $t(4)$ distinct ways. Clearly, there are $\binom{5}{4} t(4)$ distinct degenerated SLOCC classes. This situation can be considered as that five balls are divided into two groups. The first group contains exactly one ball and the second group contains exactly 4 balls. Thus, there are $\frac{5!}{1!4!}$ different ways[21]. Note that four balls correspond to four qubits. Therefore, for the case, the number of the degenerated SLOCC classes can be rewritten as $\frac{5!}{1!4!}t(1)t(4)$. We can consider this case as a partition of 5: $1 + 4 = 5$.

Case 2. Only 3-qubit true entanglement accompanied with two separable qubits

For example, $ABC - D - E$, where $ABC$ is truly entangled. As indicated in [4], three qubits can truly be entangled in two inequivalent ways. It is easy to see that there are $2\binom{5}{3}$ distinct degenerated SLOCC classes. Let us consider that five balls are divided into three groups. The first two groups contain exactly one ball respectively and the third group contains exactly three balls. Thus, there are $\frac{5!}{1!1!3!}$ different ways[21]. Note that $ABC - D - E$ and $ABC - E - D$ represent the same class. Hence, for the case, the number of the degenerated

SLOCC classes can be rewritten as $\frac{5!}{1!1!3!} * t(1)t(1)t(3) * \frac{1}{2!}$. We can remember this case as a partition of 5: $1 + 1 + 3 = 5$.

Case 3. Only 2-qubit true entanglement accompanied with three separable qubits

For example, $AB-C-D-E$, where $AB$ is a two-qubit $GHZ$ state. It is clear that there are $\binom{5}{2}$ distinct classes. We consider that five balls are divided into four groups. The first three groups contain exactly one ball respectively and the fourth group contains exactly two balls. Thus, there are $\frac{5!}{1!1!1!2!}$ different ways[21]. Note that $AB - X - Y - Z$, where $X$, $Y$ and $Z$ is any one of 3! permutations of $C$, $D$ and $E$, represent the same class. Then, for the case, the number of the degenerated SLOCC classes can be rewritten as $\frac{5!}{1!1!1!2!} * t(1)t(1)t(1)t(2) * \frac{1}{3!}$. Let us consider this case as a partition of 5: $1+1+1+2 = 5$.

Case 4. Two $GHZ$ pairs accompanied with a separable qubit

For example, $AB - CD - E$, where $AB$ and $CD$ are two-qubit $GHZ$ states. For this case, there are 15 degenerated SLOCC classes. Note that $AB-CD-E$ and $CD - AB - E$ represent the same class. Similarly, for the case, the number of the degenerated SLOCC classes can be rewritten as $\frac{5!}{1!2!2!} * t(1)t(2)t(2) * \frac{1}{2!}$. We can consider this case as a partition of 5: $1 + 2 + 2 = 5$.

Case 5. two-qubit $GHZ\otimes$ three-qubit $GHZ$ and two-qubit $GHZ\otimes$ three-qubit $W$

For example, $AB - CDE$. We can list the entanglement ways as follows.

$AB-CDE$; $AC-BDE$; $AD-BCE$; $AE-BCD$; $BC-ADE$; $BD-ACE$; $BE - ACD$; $CD - ABE$; $CE - ABD$; $DE - ABC$.

Hence, there are $2\binom{5}{2}$ degenerated SLOCC classes. Similarly, for the case, the number of the degenerated SLOCC classes can be rewritten as $\frac{5!}{2!3!} * t(2)t(3)$. We can remember this case as a partition of 5: $2 + 3 = 5$.

Case 6. A product state: $A - B - C - D - E$

It is trivial that $\frac{5!}{1!1!1!1!1!} * t(1)t(1)t(1)t(1)t(1) * \frac{1}{5!} = 1$. Let us consider this case as a partition of 5: $1 + 1 + 1 + 1 + 1 = 5$.

In total, there are $5 * t(4) + 66$ degenerated SLOCC classes for five-qubits.

2. **The exact recursive formula of $d(n)$**

For $n-$qubits, we consider the degenerated entanglement way $r_1 \otimes r_2 \otimes ... \otimes r_k$, where $k \geq 2$ and the $r_i$ qubits are truly entangled, $i = 1, 2, ..., k$. As indicated in [21], there are $\frac{n!}{r_1!r_2!..r_k!}$ different ways to divide $n$ balls into $k$ groups such that the $j$th group contains exactly $r_j$ balls, where $r_1 + r_2 + ... + r_k = n$. Note that $r_i$ qubits can truly be entangled in $t(r_i)$ distinct ways. Let $s_j$ be the number of the concurrences of $r_{i_j}$ in $r_1, r_2, ..., r_k$. Thus, for the situation, there are $\frac{n!}{r_1!r_2!..r_k!} t(r_1)t(r_2)...t(r_k) \frac{1}{s_1!s_2!...s_l!}$ degenerated SLOCC classes. In total, $d(n) = \sum \frac{n!}{r_1!r_2!..r_k!} t(r_1)t(r_2)...t(r_k) \frac{1}{s_1!s_2!...s_l!}$, where the summation is extended over all the following Euler's partitions of $n$: $r_1 + r_2 + ... + r_k = n$ in which $k \geq 2$ and $1 \leq r_1 \leq r_2 \leq ... \leq r_k < n$.

3. **Compute $d(4)$ using the recursive formula**

For four-qubits, the following are the partitions of 4.

$1 + 3$; $1 + 1 + 2$; $2 + 2$; $1 + 1 + 1 + 1$.

Case 1. For the partition 1+3, there are $\frac{4!}{1!3!} t(1)t(3) = 8$ degenerated SLOCC

classes. They are $A-BCD$, $B-ACD$, $C-ABD$, $D-ABC$. Note that three qubits can truly be entangled in two inequivalent ways.

Case 2. For the partition $1+1+2$, there are $\frac{4!}{1!1!2!}t(1)t(1)t(2)\frac{1}{2!}=6$ degenerated SLOCC classes. They are $A-B-CD$, $A-C-BD$, $A-D-BC$, $B-C-AD$, $B-D-AC$, $C-D-AB$.

Case 3. For the partition $2+2$, there are $\frac{4!}{2!2!}t(2)t(2)\frac{1}{2!}=3$ degenerated SLOCC classes. They are $AB-CD$, $AC-BD$, $AD-BC$.

Case 4. For the partition $1+1+1+1$, this case is a product state. It is trivial that $\frac{4!}{1!1!1!1!}*t(1)t(1)t(1)t(1)*\frac{1}{4!}=1$.

Totally, there are 18 degenerated SLOCC classes. See [9].

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